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On m-Open Sets and Some Classes of Functions in Generalized Topological Spaces

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On μ – Open Sets and Some Classes of Functions in Generalized Topological Spaces

استكمالاً لمتطلبات الحصول على درجة الماجستير في الرياضيات.

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Dedication

To

My mother my mother my mother took of you and

put and fondle and stayed and calledand deposited until I became

My mother met my mother and me to you and all the companionship, love, obedience

Dedicate my success to you my mother, who are accompanied by

To

My father has paid off and your branches bloomed Dauatc ask God to bless and provideyour age.

To

My Brothers.

Mohammad Jaber Salem Al-Orjan

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Also, I render my sincere thanks to members of my committee

Mohammad Jaber Salem Al-Orjan

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List of Symbols

Int (A)	the interior of a subset A in the toplogy t.
Cl(A)	the closure of a subset A in the toplogy t.
PO(X)	the family of all preopen subset of a space (X, t).
PC(X)	the family of all preclosed subset of a space (X, t).
SO(X)	the family of all semi-open subset of a space (X, t).
EX	extremally disconnected of a space (X, t).
(X, m)	generalized topological space.
exp$X$	the power set of X.
μ-PO(X)	the family of all preopen subset of a space (X, m).
μ-SO(X)	the family of all semi-open subset of a space (X, m).
μ-β-O(X)	the family of all μ-β-open subset of a space (X, m).
μ-α-O(X)	the family of all μ-α-open subset of a space (X, m).
σ	the family of all μ-semi-open i.e. $A \subseteq c_m(i_m(A))$.
π	the family of all μ-preopen i.e. $A \subseteq i_m(c_m(A))$.
α	the family of all μ-α-open i.e. $A \subseteq i_m(c_m(i_m(A)))$.
β	the family of all μ-β-open i.e. $A \subseteq c_m((i_m(c_m(A)))$.
$i_m(A)$	the interior of a subset A in (X, m).
$c_m(A)$	the closure of a subset A in (X, m).
μ-ker(A)	the μ-kernal of A in (X, m).
M_m	the union of all μ-open sets in (X, m).
μ-dense(A)	the μ-dense of A in (X, m).

Abstract

On μ-Open Sets and Some Classes of Functions in Generalized Topological Spaces

Mohammad Jaber Al-orjan

Mu'tah University, 2012

In this thesis, we investigated and study some classes of function called almost continuous function, weak continuous function, generalized continuous function defined by generalized open sets and contra continuous function. Also we introduce and investigate the notions of weakly (m,l)-open functions and weakly (m,l)-closed functions, totally (m,l)-continuous functions, slightly (m,l)-continuous functions and study some of the fundamental properties of this class of functions.

الملخـــص

حول المجموعات المفتوحة المعممة وبعض أصناف الاقترانات باستخدامها

محمد جبر سالم العرجان

جامعة مؤته، 2012

في هذه الرسالة درسنا وبحثنا في بعض صفوف الاقترانات والتـــي تـــسمى الاقترانات المتصلة تقريبا، الاقترانات المتـــصلة الـــضعيفة، الاقترانـــات المتـــصلة المعممة المعرفة بالمجموعات المفتوحة المعممة و ضد الاقترانات المتصلة. وكـــذلك قدمنا وبحثنا مفاهيم الاقترانات (m, l) المفتوحة الضعيفة، الاقترانات (m, l) المغلقة الضعيفة، الاقترانات (m, l) المتصلة كليا والاقترانات (m, l) المتصلة قليلا ودرسنا بعض الخواص الأساسية لهذا الصف من الاقترانات.

Chapter One
Introduction and Preliminaries

In this chapter, we introduce some definitions and theorems which are needed later on in this thesis.

Topology has now become an independent science as one important branch of Mathematics branches. In addition to this it serves all other mathematical branches that are now concerned more with Topology and its application processes. Topology science comes from the study of Limits and Continuity. 'Topology' is derived from the Latin origin of the word which means: 'place/ site/ or blank,' and the first user of the word is the German mathematician Lesting in 1847, in order to come up with Topology and Site Engineering out of modern mathematical theories and structures that were founded in the nineteenth century and modified in the twentieth century. Throughout its development, Topology has become a totally independent science that is now used as a tool serving the whole science of Mathematics. It also can be said, as a matter of making things easier, that this science is concerned with mathematical characteristics that never get affected when transferring from one mathematical atmosphere to another. Topology also suggests certain scientific theories and tries proving them. In other words, it is almost similar to the traditional engineering science.

In 1978, A. Csaszar, introduce the Generalized topological spaces.
In 2002, A. Csaszar, introduce the Generalized topology, Generalized continuity. In 2009, W.K. Min, introduce the Almost continuity on generalized topological spaces. In 2009, W.K. Min, introduce the Weak continuity on generalized topological spaces. In 2010, W.K. Min, introduce the Generalized continuous functions defined by generalized open sets on generalized topological spaces. In 2012, A. AL-Omari, and T. Noiri, introduce the contra continuous function in generalized topological spaces.

1.1 Introduction:

In [A. Csaszar, 1998]. A. Csaszar founded the theory of generalized topological spaces, and studied the elementary character of these classes. Especially he introduced the notions of continuous functions on generalized topological spaces, and investigated characterizations of generalized continuous functions $(=(m,l)$-continuous functions in [A. Csaszar, 2002]. We recall some notions defined in [A. Csaszar, 2002] . Let X be a non-empty set and *exp* X the power set of X. We call a class $m \subseteq \exp X$ a generalized topology [A. Csaszar, 2002] (briey, GT) if $f \in m$ and the arbitrary union of elements of μ belongs to μ. A set X

with a GT μ on it is called a generalized topological space (briey, GTS) and is denoted by (X, m). For a GTS (X, m), the elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets.

Definition 1.1.1. Let (X, t) be a topological space. A subset A of *X*. Is said to be

(1) semi-open [N. Levine, 1963] if $A \subset cl(\mathrm{int}(A))$;

(2) preopen [A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, 1982] if $A \subset \mathrm{int}(cl(A))$;

(3) *α*-open [O. Njastad, 1965] if $A \subset \mathrm{int}(cl(\mathrm{int}(A)))$;

(4) β-open [M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, 1983] if $A \subset cl(\mathrm{int}(cl(A)))$;

(5) b-open [D. Andrijevic, 1996] if $A \subset cl(\mathrm{int}(A)) \cup \mathrm{int}(cl(A))$).

Example 1.1.2. Let $X = \{a,b,c,d\}$. Consider the generalized topologies $m = \{f, \{a\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$. μ is a generalized topological since $f \in m$ and the arbitrary union of elements of μ belongs to μ.

For $A \subseteq X$, we denote by $c_m(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A; (or $c_m(A)$ = X if there is no such μ-closed subset of A). And by $i_m(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A. (or $i_m(A)$=Ø if there is no μ -open set contained in *A*).

Example 1.1.3. Let $X = \{a,b,c\}$ and $m = \{f, \{a\}, \{b\}, \{a,b\}\}$ a GT on *X*. $i_m(\{a,b\}) = \{a,b\}$ since the largest μ-open set contained in $\{a,b\}$ is $\{a,b\}$. And $c_m(\{a,b\}) = \{a,b,c\} = X$ since the smallest μ-closed containing $\{a,b\}$ is X. We observe that the set of all μ-closed is $\{X, \{b,c\}, \{a,c\}, \{c\}\}$.

In this thesis, we investigated and study some classes of functions called Almost continuous functions, weak continuous functions, generalized continuous functions defined by generalized open sets and contra continuous functions. Also we introduce and investigate the notions of weakly (m, l)-open functions and weakly (m, l)-closed functions and study some of the fundamental properties of this class of functions. Moreover we introduce the totally (m, l)-continuous functions and

strongly (m,l)-continuous functions and slightly (m,l)-continuous functions and (m,l)-continuous functions.

This thesis consists of three chapters. In chapter one, definition and theorems that are needed later in this thesis are given. In chapter two, we introduce basic definition of continuous function, almost continuous functions , weakly continuous functions, generalized continuous functions defined by generalized open sets and contra continuous functions. Moreover some important theorems, properties and examples.

In chapter three, we introduce and investigate the notions of weakly (m,l)-open functions , weakly(m,l)-closed functions, totally (m,l)-continuous functions, strongly (m,l)-continuous functions, slightly (m,l)-continuous functions and (m,l)-continuous functions. And study some of the fundamental properties of this class of functions.

1.2 Preliminaries

Theorem 1.2.1. [W.K.Min, 2009]**.** Let (X,m) be a generalized topological space and $A \subseteq X$. Then

(1) $x \in c_m(A)$ if and only if $M \cap A \neq f$ for each $M \in m$ containing x.

(2) $x \in i_m(A)$ if and only if there exists $M \in m$ such that x$\in$ $M \subseteq A$.

Proof. (1).**(⇒).** Suppose that $x \in c_m(A)$ and M is μ-open such that

$x \in M \in m$ assume that $M \cap A = f$. Now $A \subseteq X - M$ since X-M is μ-closed then by theorem $c_m(A) \subseteq X - M$. But $x \in c_m(A)$ this implies that $x \notin M$. Contradiction hence $M \cap A \neq f$.

(⇐). Assume $x \notin c_m(A)$ this means that $x \in X - c_m(A)$ by

assumption $(X - c_m(A)) \cap A \neq f$ this means that there exists $a \in X - c_m(A)$ and $a \in A$ this means that $a \notin c_m(A)$ but $a \in A \subseteq c_m(A)$ then this implies that $a \in c_m(A)$ this is a contradiction. Hence $x \in c_m(A)$.

(2) Let $x \in i_m(A)$ this implies that $x \notin X - i_m(A)$ lead to $x \notin c_m(X - A)$ there exists $M \in m$ and $x \in M$ such that $M \cap (X - A) = f$ this implies that $M \subseteq A$.

($\Leftarrow$). The proof is obvious.

Theorem 1.2.2. [W.K.Min, 2009]. Let (X, m) be a generalized topological space on X. If $A \subseteq F$ and F is μ-closed then $c_m(A) \subseteq F$.

Proof. Since F is one of the members of the intersection this implies that the intersection subset of F this implies that $c_m(A) \subseteq F$.

Theorem 1.2.3. [A.Csaszar, 2002]. Let (X, m) be generalized topological spaces. Then

(1) $c_m(A) = X - i_m(X - A)$.

(2) $i_m(A) = X - c_m(X - A)$.

Proof. **(1)** $X - i_m(X - A) = X - \cup\{F \subseteq X : X - A \subseteq F,\ F \in m\}$

$= \cap\{X - F \subseteq X : X - F \subseteq A,\ F \in m\}$

$= c_m(A)$

(2) $X - c_m(X - A) = X - \cap\{F \subseteq X : X - A \subseteq F,\ F \in m\}$

$= \cup\{X - F \subseteq X : X - F \subseteq A,\ F \in m\}$

$= i_m(A)$

Theorem 1.2.4. [A.Csaszar, 1998]. Let (X, m) be a generalized topological space on X. If $V \subseteq A$ and V is μ-open then $V \subseteq i_m(A)$.

Proof. Suppose that $A \subseteq V$ and V is μ-open**.** Let $x \in V$ since $V \subseteq A$ this implies that $x \in V \subseteq A$ this implies that $x \in i_m(A)$. Hence $V \subseteq i_m(A)$**.**

Theorem 1.2.5. [A.Csaszar, 1998]. Let (X, m) be a generalized topological space on X. If $A \subseteq B$ then

(1) $c_m(A) \subseteq c_m(B)$.

(2) $i_m(A) \subseteq i_m(B)$.

Proof. (1) Suppose $A \subseteq B$. now $A \subseteq B \subseteq c_m(B)$ this implies that $A \subseteq c_m(B)$ but $c_m(B)$ is μ-closed and $A \subseteq c_m(B)$ by Theorem 1.2.2. $c_m(A) \subseteq c_m(B)$.

(2) Suppose $A \subseteq B$, now $i_m(A) \subseteq A \subseteq B$ this implies that $i_m(A) \subseteq B$ but $i_m(A)$ is μ-open and $i_m(A) \subseteq B$ by Theorem1.2.4. $i_m(A) \subseteq i_m(B)$.

Theorem 1.2.6. [A.Csaszar, 1998]. Let (X, m) be a generalized topological space on X. and $A \subseteq X$ then

(1) A is μ-closed if and only if $A = c_m(A)$.

(2) A is μ-open if and only if $A = i_m(A)$.

Proof. The proof is obvious.

Theorem 1.2.7. [A.Csaszar, 1998]. Let (X, m) be a generalized topological space on X. And $A \subseteq X$ then

(1) $c_m(c_m(A)) = c_m(A)$.

(2) $i_m(i_m(A)) = i_m(A)$.

Proof. (1) Since $c_m(A)$ is μ-closed by Theorem 1.2.6. $c_m(c_m(A)) = c_m(A)$.

(2) We want to prove $i_m(i_m(A)) \subseteq i_m(A)$ and $i_m(A) \subseteq i_m(i_m(A))$. $i_m(i_m(A)) \subseteq i_m(A)$ is true since $i_m(A) \subseteq (A)$.

The $i_m(A) \subseteq i_m(i_m(A))$. Let $x \in i_m(A)$ there exists μ-open set $M \in m$ such that $x \in M \subseteq i_m(A)$. Now $M \subseteq i_m(A)$ and M is μ-open by Theorem 1.2.4. $i_m(A) \subseteq i_m(i_m(A))$. Hence $i_m(i_m(A)) = i_m(A)$.

Definition 1.2.8. [A. Csaszar, 2005]. Let (X, m) be a generalized topological space and $A \subseteq X$. Then A is said to be:

(1) μ-semi-open if $A \subseteq c_m(i_m(A))$, (denoted by σ).

(2) μ-preopen if $A \subseteq i_m(c_m(A))$, (denoted by π).

(3) μ-*α*-open if $A \subseteq i_m(c_m(i_m(A)))$, (denoted by α).

(4) μ-β-open if $A \subseteq c_m((i_m(c_m(A)))$, (denoted by β).

(5) μ-regular-open if $A = i_m(c_m(A))$.

The complement of a μ-semi-open (μ-SO(X)) set (resp. μ-preopen μ-PO(X), μ-α-open (μ-α-O(X), μ-β-open (μ-β-O(X)), μ-regular-open) is said to be μ-semi-closed (resp. μ-preclosed, μ-α- closed, μ-β-closed, μ-regular-closed).

Theorem 1.2.9. [W.K. Min, 2009]. Let (X, m) be a generalized topological space and $A \subseteq X$. A is

(1) μ-semiclosed if and only if $i_m(c_m(A)) \subseteq A$,

(2) μ-preclosed if and only if $c_m(i_m(A)) \subseteq A$,

(3) μ-reguler closed if and only if $A = c_m(i_m(A))$,

(4) μ-β -closed if and only if $i_m(c_m(i_m(A))) \subseteq A$.

Proof. Take the complementary and the proof is obvious.

Theorem 1.2.10. [W.K. Min, 2009]**.** Let (X, m) be a generalized topological space and $A \subseteq X$. Then X is both a μ-semi-open set and a μ-β-open set .

Proof. Suppose that $i_m(X) \subseteq c_m(i_m(X)) = F \neq X$. Then $f \neq X - F$ and X-F is a μ-open . Since $i_m(X)$ is the largest μ-open set in X, it has to contain X-F. So it is impossible that F contains $i_m(X)$: a contradiction. Hence $X = c_m(i_m(X))$, that is, X is μ-semiopen. Similarly, X is also a μ-β-open set since $c_m(X) = X$.

Lemma 1.2.11. [A. AL-Omari, and T. Noiri , 2012]. For a subset A of a GTS (X, m), the following properties are equivalent :

(1) A is μ-regular-closed;
(2) A is μ-preclosed and μ-semi-open;
(3) A is μ-α-closed and μ-β-open.

Proof. (1) ⇒ (2). Let A be μ-regular-closed. Then $A = c_m(i_m(A))$ and A is μ-preclosed and μ-semi-open.

(2) ⇒ (3). Let A be μ-preclosed and μ-semi-open. Then $A \subseteq c_m(i_m(A))$ and $c_m(i_m(A)) \subseteq A$. Therefore, we have $c_m(A) = c_m(i_m(A))$ and hence $c_m(i_m(c_m(A))) = c_m(i_m(c_m(i_m(A))))$ $= c_m(i_m(A)) \subseteq A$. This shows that A is μ-α-closed. Since $s(m) \subseteq b(m)$, it is obvious that A is μ-β-open.

(3) ⇒ (1). Let A be μ-α-closed and μ-β-open.

Then $A = c_m(i_m(c_m(A)))$ and hence $c_m(i_m(A)) = c_m(i_m(c_m(i_m(c_m(A)))))$ $= c_m(i_m(c_m(A))) = A$. Therefore, A is μ- regular -closed.

Chapter Two
Continuity On Generalized Topological Spaces

2.1 (m, l)-Continuous Function.

In this section we study the (m, l)-continuity on generalized topological spaces.

Definition 2.1.1. [A. Csaszar, 2002]. Let (X, m) and (Y, l) be two generalized topological spaces. Then a function $f : X \to Y$ is said to be (m, l)-continuous if $A \in l$ implies that $f^{-1}(A) \in m$.

Example 2.1.2. Let $X = \{a, b, c, d\}$ consider two generalized topologies $m_1 = \{f, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $m_2 = \{f, \{c\}, \{b, c\}\}$ on X. Define $f : (X, m_1) \to (X, m_2)$ as follows: $f(a) = b$, $f(b) = f(d) = d$, $f(c) = c$. Then f is (m_1, m_2)-continuous. Since $f^{-1}(f) = f$, $f^{-1}(\{c\}) = \{c\}$ and $f^{-1}(\{b, c\}) = \{a, c\}$.

Theorem 2.1.3. [W.K. Min, 2009. W.K. Min, 2009]. Let $f : X \to Y$ be a function between a GTS's (X, m) and (Y, l). Then the following statements are equivalent:

(1) f is (m, l)-continuous.

(2) $f^{-1}(i_l(B)) \subseteq i_m(f^{-1}(B))$ for all $B \subseteq Y$.

(3) $c_m(f^{-1}(B) \subseteq f^{-1}(c_l(B))$ for all $B \subseteq Y$.

(4) $f(c_m(A)) \subseteq c_l(f(A))$ for all $A \subseteq X$.

(5) For every l-closed set F in Y, $f^{-1}(F)$ is μ-closed in X.

(6) For each $x \in X$ and each l-open set V containing f(x), there exist a μ-open set U containing x such that $f(U) \subseteq V$.

Proof . (1) ⇒ (2). Let f be (m, l)-continuous and, $B \subseteq Y$ since $i_l(B)$ is l-open in Y, by (m, l)-continuity $f^{-1}(i_l(B))$ is μ-open in X. So $i_m(f^{-1}((i_l)B)) = f^{-1}(i_l(B))$ ....(1)

Now $i_l(B) \subseteq (B)$ this implies $f^{-1}(i_l(B)) \subseteq f^{-1}(B)$ take the μ-interior to both sides $i_m(f^{-1}(i_l(B))) \subseteq i_m(f^{-1}(B))$ by (1). Thus $i_m(f^{-1}(B)) \supseteq f^{-1}(i_l(B))$.

(2) ⇒ (1). Let $f^{-1}(i_l(B)) \subseteq i_m(f^{-1}(B))$ for all $B \subseteq Y$.

Let H be any l-open subset of Y this implies $i_l(H) = (H)$ by hypothesis $f^{-1}(i_l(H)) \subseteq i_m(f^{-1}(H))$ since $i_l(H) = (H)$ this implies

$f^{-1}(H)=i_m(f^{-1}(H))$ and we obtain $f^{-1}(H)$ is μ-open. Thus f is (m,l)-continuous.

(2)$\Leftrightarrow$(3). It follows from Theorem 1.2.3.

(3)$\Leftrightarrow$(4). By (3) , $c_m\left(f^{-1}\left(f(A)\right)\right)\subseteq f^{-1}\left(c_l\left(f(A)\right)\right)$ for $A\subseteq X$. Hence we have $f\left(c_m(A)\right)\subseteq c_l\left(f(A)\right)$. Similarly, from (4) we have (3).

(1) $\Rightarrow$ (5). Let f is (m,l)-continuous, and F is l-closed in Y, then $Y-F$ is l-open in Y. Since f is (m,l)-continuous then $f^{-1}(Y-F)$ is μ-open. But $f^{-1}(Y-F)=f^{-1}(Y)-f^{-1}(F)=X-f^{-1}(F)$. Since $X-f^{-1}(F)$ is μ-open this implies that $f^{-1}(F)$ is μ-closed.

(5) $\Rightarrow$(1). Let $f^{-1}(F)$ is μ-closed in X and let H be any l-open in Y. Then (Y-H) is l-closed then $f^{-1}(Y-H)$ is l-closed. But $f^{-1}(Y-H)=f^{-1}(Y)-f^{-1}(H)=X-f^{-1}(H)$ then $X-f^{-1}(H)$ is l-closed then $f^{-1}(H)$ is μ-open. Hence f is (m,l)-continuous.

(1) $\Rightarrow$(6). Let $x\in X$ and l-open set V containing f(x), since f is (m,l)-continuous then $f^{-1}(V)$ is μ-open set in X such that $x\in f^{-1}(V)$. Now assume $f^{-1}(V)=U$ then there exists a μ-open set U containing x such that $f(U)\subseteq V$.

(6) $\Rightarrow$(1). Let V be a l-open set in Y and $x\in f^{-1}(V)$. Then by (6), There exist a μ-open set U_x containing x such that $f(U_x)\subseteq V$. And hence $f^{-1}(V)=\bigcup U_X$ which is μ-open.

2.2. Almost (m,l)-Continuous Functions on Generalized Topological Spaces.

In this section we study the notion of almost (m,l)-continuous functions on GTS's and investigate properties of such functions and relationships among (m,l)-continuity and almost (m,l)-continuity.

Definition 2.2.1. [W.K. Min,2009]. Let (X,m) *and* (Y,l) be generalized topological spaces. Then a function $f:(X,m)\to(Y,l)$ is said to be almost (m,l)-continuous at $x\in X$ if for each l-open set V containing *f*(*x*), there exists a m-open set U containing x such that $f(U)\subseteq i_l(c_l(V))$. And is said to be almost (m,l)-continuous if it is almost (m,l)- continuous at every point of X.

Remark 2.2.2. [W.K. Min, 2009]. From the above definition of almost (m,l)-continuity, we have the following implication but the reverse relation may not be true in general:

(m,l)-continuous $\Rightarrow$ almost (m,l)-continuous.

Example 2.2.3. Let $X=\{a,b,c,d\}$ consider two generalized topologies $m_1=\{f,\{a\},\{a,b\},\{b,c\},\{a,b,c\}\}$ and $m_2=\{f,\{a,c\},\{b,c\},\{a,b,c\}\}$. On X. Define $f:(X,m_1)\rightarrow(X,m_2)$ as follows:

$$f(a)=b,\ f(b)=f(d)=d,\ f(c)=c.$$

Then f is almost (m_1,m_2)-continuous but it is not (m_1,m_2)-continuous. Since $A=\{a,c\}\in m_2$ and $f^{-1}(A)=\{c\}\notin m_1$

Theorem 2.2.4. [W.K. Min,2009]. Let $f:X\rightarrow Y$ be a function on the generalized topological spaces (X,m) *and* (Y,l). Then the following are equivalent:

(1) f is almost (m,l)-continuous at $x\in X$.

(2) $x\in i_m(f^{-1}(i_l(c_l(V))))$ for every l-open subset V containing $f(x)$.

(3) $x\in i_m(f^{-1}(V))$ for every l-regular-open subset V containing $f(x)$.

(4) For any l-regular set V containing $f(x)$, there exists a m-open set U containing x such that $f(U)\subseteq V$.

Proof. (1) $\Rightarrow$ (2). Let V be any l-open subset of Y containing $f(x)$. From the definition of almost continuity, there exists a m-open subset U of X containing x such that $f(U)\subseteq i_l(c_l(V))$. Since $x\in U\subseteq f^{-1}(i_l(c_l(V)))$, we have $x\in i_m(f^{-1}(i_l(c_l(V))))$.

(2) $\Rightarrow$ (3). Let V be any l-regular-open subset of Y containing $f(x)$. Then since $V=i_l(c_l(V))$, by (2), we have $x\in i_m(f^{-1}(V))$.

(3) $\Rightarrow$ (4). Let V be any l-regular-open subset of Y containing $f(x)$. From (3), there exists a m-open set U containing x such that $U\subseteq f^{-1}(V)$. And we have $f(U)\subseteq V$.

(4) $\Rightarrow$ (1). Let V be any l-open subset of Y containing $f(x)$. Then $f(x)\in V\subseteq i_l(c_l(V))$. Since $i_l(c_l(V))$ is l-regular-open, by (4) there exists a m-open set U containing x such that $f(U)\subseteq i_l(c_l(V))$. Hence f is almost (m,l)- continuous.

Theorem 2.2.5. [W.K. Min,2009]. Let $f:X\rightarrow Y$ be a function on the generalized topological spaces (X,m) *and* (Y,l). Then the following are equivalent:

(1) f is almost (m,l)-continuous.

(2) $f^{-1}(V)\subseteq i_m(f^{-1}(i_l(c_l(V))))$ for every l-open subset V in Y.

(3) $c_m(f^{-1}(c_l(i_l(F))))\subseteq f^{-1}(F)$ for every l-closed subset F in Y.
(4) $c_m(f^{-1}(c_l(i_l(c_l(B)))))\subseteq f^{-1}(c_l(B))$ for every subset B in Y.
(5) $f^{-1}(i_l(B))\subseteq i_m(f^{-1}(i_l(c_l(i_l(B)))))$ for every subset B in Y.
(6) For every l-regular-open subset V in Y, $f^{-1}(V)$ is m-open.
(7) For every l-regular-closed subset F in Y, $f^{-1}(F)$ is m-closed.

Proof: (1) ⇒ (2). Let V be a l-open set in Y and $x\in f^{-1}(V)$. There exists a m-open set U of X containing x such that $f(U)\subseteq i_l(c_l(V))$. This implies $x\in i_m(f^{-1}(i_l(c_l(V))))$. Hence

$$f^{-1}(V)\subseteq i_m(f^{-1}(i_l(c_l(V)))).$$

(2) ⇒ (3). Let F be a l-closed set in Y. From Theorem 1.2.3**,** it follows

$$\begin{aligned} f^{-1}(Y-F) &\subseteq i_m(f^{-1}(i_l(c_l(Y-F)))) \\ &= i_m(f^{-1}(Y-c_l(i_l(F)))) \\ &= X-c_m(f^{-1}(c_l(i_l(F)))). \end{aligned}$$

Hence we have $c_m(f^{-1}(c_l(i_l(F))))\subseteq f^{-1}(F)$.

(3) ⇒ (4). Since $c_m(B)$ is l-closed in Y by (3) we have (4).

(4) ⇒ (5). It follows from Theorem 1.2.3

(5) ⇒ (6). Let V be any l-regular-open set of Y.
Since $i_l(c_l(i_l(V)))=V$, from (5), it follows $f^{-1}(V)\subseteq i_m(f^{-1}(V))$ and hence $f^{-1}(V)\subseteq i_m(f^{-1}(V))$**.**

(6) ⇒ (7). Let F be any l-regular-closed set of Y. Then by (6), we have $X-f^{-1}(F)=f^{-1}(Y-F)=i_m(f^{-1}(Y-F))=X-c_m(f^{-1}(F))$.
is μ-open and hence $f^{-1}(F)$ is μ-closed.

(7)⇒(1). Let V be any l-regular-open set of Y containing $f(x)$. By (7), $X-f^{-1}(V)=f^{-1}(Y-V)=c_m(f^{-1}(Y-V))=X-i_m(f^{-1}(V))$.
Since $x\in f^{-1}(V)=i_m(f^{-1}(V))$, there exists a m-open set U containing x such that $U\subseteq f^{-1}(V)$. Hence by Theorem 2.2.4. (4), f is almost (m,l)-continuous.

Theorem 2.2.6. [W.K. Min, 2009]. Let $f:X\rightarrow Y$ be a function on the generalized topological spaces (X,m) *and* (Y,l). Then the following statements are equivalent:
(1) f is almost (m,l)-continuous.
(2) $c_m(f^{-1}(G))\subseteq f^{-1}(c_l(G))$ for every l-β-open set G of Y.

(3) $c_m(f^{-1}(G)) \subseteq f^{-1}(c_l(G))$ for every l-semiopen set G of Y.
(4) $f^{-1}(V) \subseteq i_m(f^{-1}(i_l(c_l(V))))$ for every l-preopen set V of Y.

Proof. (1) ⇒ (2). Let G be any l-β-open set. Since $c_l(G)$ is l-regular-closed, by Theorem 2.2.5. (7), $c_m(f^{-1}(c_l(G))) = (f^{-1}(c_l(G)))$.
Thus $c_m(f^{-1}(G)) \subseteq c_m(f^{-1}(c_l(G))) = f^{-1}(c_l(G)))$.

(2) ⇒ (3). It is obvious since every l-semiopen set is l-β-open.

(3) ⇒ (1). Let F be any l-regular-closed set of Y, then since F is l-semiopen, we have $c_m(f^{-1}(F)) \subseteq f^{-1}(c_l(F)) = f^{-1}(F)$. Thus from Theorem 2.2.5. (7), f is almost (m,l)-continuous.

(1) ⇒ (4). Let V be any l-preopen set of Y; then $V \subseteq i_l(c_l(V))$ and $i_l(c_l(V))$ is l-regular-open. By Theorem 2.2.5. (6),
$f^{-1}(i_l(c_l(V))) = i_m(f^{-1}(i_l(c_l(V))))$. Thus we have
$f^{-1}(V) \subseteq f^{-1}(i_l(c_l(V))) = i_m(f^{-1}(i_l(c_l(V))))$.

(4)⇒ (1). Let V be any l-regular-open set of Y, then V is l-preopen and $f^{-1}(V) \subseteq i_m(f^{-1}(i_l(c_l(V)))) = i_m(f^{-1}(V))$.
Hence by Theorem 2.2.5. (6), f is almost (m,l)-continuous.

2.3 Weakly (m,l)-Continuous Functions

In this section we study the notion of weak (m,l)-continuity on generalized topological spaces, and investigate characterizations for such functions.

Definition 2.3.1. [W.K. Min,2009]. Let (X,m) *and* (Y,l) be generalized topological spaces. Then a function $f: X \to Y$ said to be weakly (m,l)-continuous if for each $x \in X$ and each l-open set V containing $f(x)$, there exists a m-open set U containing x such that $f(U) \subseteq c_l(V)$.

Remark 2.3.2. The following implication is true but the reverse relation may not be true in general:

$$(m,l)\text{-continuous} \Rightarrow \text{weakly } (m,l)\text{-continuous.}$$

Example 2.3.3. Let $X = \{a,b,c,d\}$. Consider two generalized topologies $m_1 = \{f,\{a\},\{a,b\},\{b,c\},\{a,b,c\}\}$ and $m_2 = \{f,\{a,c\},\{b,c\},\{a,b,c\}\}$. On X. Define $f:(X,m_1) \to (X,m_2)$ as follows:
$f(a) = b$, $f(b) = f(d) = d$, $f(c) = c$. Then since

$c_{m_2}(\{a,c\})=c_{m_2}(\{b,c\})=X$ it is that f is weakly (m_1,m_2)-continuous. But f is not (m_1,m_2)-continuous. Since $f^{-1}(\{a,c\})=\{c\}\notin m_1$

Remark 2.3.4. The following implications is true but the reverse relations may not be true in general:

(m,l)-continuous $\Rightarrow$ almost (m,l)-continuous $\Rightarrow$ weakly (m,l)-continuous

Example 2.3.5. (1) Let $X=\{a,b,c,d\}$. Consider two generalized topologies $m_1=\{f,\{a\},\{a,b\},\{b,c\},\{a,b,c\}\}$ and $m_2=\{f,\{a,c\},\{b,c\},\{a,b,c\}\}$ on X. Define $f:(X,m_1)\to(X,m_2)$ as follows: $f(a)=b$, $f(b)=f(d)=d$, $f(c)=c$. Then f is almost (m_1,m_2)-continuous but it is not (m_1,m_2)- continuous.

(2) Let $X=\{a,b,c,d\}$. Consider two generalized topologies $m_1=\{f,\{a,b,c\},\{d\},X\}$ and $m_2=\{f,\{a,c\},\{a,c,d\},\{b,c,d\},\{d\},X\}$ on *X*, and the identity function $f:(X,m_1)\to(X,m_2)$. Then f is weakly (m_1,m_2)-continuous but it is not almost (m_1,m_2)-continuous.

Theorem 2.3.6. [W.K. Min,2009]. Let $f:X\to Y$ be a function on the generalized topological spaces (X,m) *and* (Y,l). Then the following are equivalent:

(1) f is weakly (m,l) -continuous.

(2) $f^{-1}(V)\subseteq i_m(f^{-1}(c_l(V)))$ for every l -open subset *V* of *Y*.

(3) $c_m(f^{-1}((i_l)A)))\subseteq f^{-1}(A)$ for every l -closed set A of *Y*.

(4) $c_m\left(f^{-1}\left(i_l\left(c_l(B)\right)\right)\right)\subseteq f^{-1}\left(c_l(B)\right)$ for every set *B* of *Y*.

(5) $f^{-1}\left(i_l(B)\right)\subseteq i_m\ (f^{-1}\left(\left(c_l(i_l(B)\right)\right)$ for every set *B* of *Y*.

(6) $c_m(f^{-1}(V))\subseteq f^{-1}(c_l(V))$ For every l -open subset *V* of *Y*.

Proof . (1) $\Rightarrow$ (2). Let V be a l -open subset of Y and $x\in f^{-1}(V)$. There exists a m-open subset U of X containing x such that $f(U)\subseteq c_l(V)$. Since $x\in U\subseteq f^{-1}(c_l(V))$, $x\in i_m(f^{-1}(c_l(V)))$. Hence $f^{-1}(V)\subseteq i_m(f^{-1}(c_l(V)))$.

(2) $\Rightarrow$ (3). Let A be a l -closed subset in Y. Then $Y-A$ is l -open in *Y* and , by (2)

$$f^{-1}(Y-A)\subseteq i_m(f^{-1}(c_l(Y-A)))$$
$$=i_m(f^{-1}(Y-i_l(A)))\subseteq X-c_m(f^{-1}(i_l(A))).$$

Thus $c_m\left(f^{-1}\left(i_l(A)\right)\right)\subseteq f^{-1}(A)$.

(3) $\Rightarrow$ (4). Let B be a subset of Y. Since $c_l(B)$ is l-closed in Y, from (3), it follows $c_m\left(f^{-1}\left(i_l\left(c_l(B)\right)\right)\right) \subseteq f^{-1}\left(c_l(B)\right)$.

(4) $\Rightarrow$ (5). Let B be a subset of Y. From(4), it follows

$$f^{-1}\left(i_l(B)\right) = X - f^{-1}\left(c_l(Y - B)\right) \subseteq X - c_m\left(f^{-1}\left(i_l\left(c_l(Y - B)\right)\right)\right)$$

$= i_m\left(f^{-1}\left(c_l\left(i_l(B)\right)\right)\right)$. Thus we have $f^{-1}\left(i_l(B)\right) \subseteq i_m\left(f^{-1}\left(\left(c_l\left(i_l(B)\right)\right)\right)\right)$

(5) $\Rightarrow$ (6). Let V be a l-open subset of Y. Suppose $x \notin f^{-1}\left(c_l(V)\right)$. Then $f(x) \notin c_l(V)$ and so there exists a l-open set U containing f(x) such that : $U \cap V = f$ and hence $c_l(U) \cap V = f$. By (5),

$x \in f^{-1}(U) \subseteq i_m\left(f^{-1}\left(c_l(U)\right)\right)$.

From the definition of l-interior, there exists a m-open set G containing x Such that $x \in G \subseteq f^{-1}\left(c_l(U)\right)$. Since $c_l(U) \cap V = f$ and $f(G) \subseteq c_l(U)$, we have $G \cap f^{-1}(V) = f$ and so $x \notin c_m\left(f^{-1}(V)\right)$.

Hence $c_m\left(f^{-1}(V)\right) \subseteq f^{-1}\left(c_l(V)\right)$.

(6) $\Rightarrow$ (1). Let $x \in X$ and V be a l-open set in Y containing $f(x)$. From $V = i_l(V) \subseteq i_l\left(c_l(V)\right)$ and (6) , we have

$x \in f^{-1}(V) \subseteq f^{-1}\left(i_l\left(c_l(V)\right)\right) = X - f^{-1}\left(c_l\left(Y - c_l(V)\right)\right)$

$\subseteq X - c_m\left(f^{-1}\left(Y - c_l(V)\right)\right) = i_m\left(f^{-1}\left(c_l(V)\right)\right)$. So there exists a m-open subset U containing x in X such that $U \subseteq f^{-1}\left(c_l(V)\right)$. Hence f is weakly (m, l)-continuous.

Theorem 2.3.7. [W.K. Min,2009]. Let $f : X \to Y$ be a function on the generalized topological spaces (X, m) *and* (Y, l). Then the following are equivalent:

(1) f is weakly (m, l)-continuous.

(2) $c_m\left(f^{-1}\left(i_m(K)\right)\right) \subseteq f^{-1}(K)$ for every λ-regular closed set K of Y.

(3) $c_m\left(f^{-1}\left(i_l\left(c_l(G)\right)\right)\right) \subseteq f^{-1}\left(c_l(G)\right)$ for every λ-β-open set G of Y.

(4) $c_m\left(f^{-1}\left(i_l\left(c_l(G)\right)\right)\right) \subseteq f^{-1}\left(c_l(G)\right)$ for every λ-semi-open set G of Y.

Proof . (1) $\Rightarrow$ (2). Let K be any λ-regular-closed set of Y. Then since $i_l(K)$ is λ-open, by Theorem 2.3.6 (6), we have $c_m\left(f^{-1}\left(i_l(K)\right)\right) \subseteq f^{-1}\left(c_l\left(i_l(K)\right)\right)$. Since K is l-reguler closed,

we have $c_m\left(f^{-1}\left(i_l(K)\right)\right)\subseteq f^{-1}\left(c_l\left(i_l(K)\right)\right) \subseteq f^{-1}(K)$.

(2) $\Rightarrow$ (3). Let G by any l-β-open set. Then
$c_l(G)\subseteq c_l\left(i_l\left(c_l(G)\right)\right)\subseteq c_l(G)$. So that $c_l(G)$ is l-regular closed.

From (2), it follows $c_m\left(f^{-1}\left(i_l\left(c_l(G)\right)\right)\right)\subseteq f^{-1}\left(c_l(G)\right)$.

(3) $\Rightarrow$ (4). Since every l-semiopen set is l-β-open, it is obvious.
(4) $\Rightarrow$ (1). Let V be any l-open subset of Y. Then from (4),
$c_m\left(f^{-1}(V)\right)\subseteq c_m\left(f^{-1}\left(i_l(c_l(V))\right)\right)\subseteq f^{-1}\left(c_l(V)\right)$.

Hence by 2.3.6 (6), f is weakly (m,l)-continuous.

Theorem 2.3.8. [W.K. Min, 2009]. Let $f: X\rightarrow Y$ be a function on the generalized topological spaces (X,m) *and* (Y,l). Then the following are equivalent:

(1) f is weakly (m,l) -continuous.

(2) $c_m\left(f^{-1}\left(i_l\left(c_l(G)\right)\right)\right)\subseteq f^{-1}\left(c_l(G)\right)$ for every l-preopen set G of Y.

(3) $c_m\left(f^{-1}(G)\right)\subseteq f^{-1}\left(c_l(G)\right)$ for every l-preopen set G of Y.

(4) $f^{-1}(G)\subseteq i_m\left(f^{-1}\left(c_l(G)\right)\right)$ for every l-preopen set G of Y.

Proof. (1) $\Rightarrow$ (2). Let G be any l-preopen set in Y. Then
$c_l(G)=c_l\left(i_l\left(c_l(G)\right)\right)$, and hence $c_l(G)$ is l-reguler closed.

From Theorem 2.3.7, it following that $c_m\left(f^{-1}\left(i_l\left(c_l(G)\right)\right)\right)\subseteq f^{-1}\left(c_l(G)\right)$.

(2) $\Rightarrow$ (3). Let G be l-preopen set, then $G\subseteq i_l\left(c_l(G)\right)$ and by (2)
$c_m\left(f^{-1}\left(i_l\left(c_l(G)\right)\right)\right)\subseteq f^{-1}\left(c_l(G)\right)$ since G is l-preopen this implies that $c_m\left(f^{-1}(G)\right)\subseteq f^{-1}\left(c_l(G)\right)$.

(3) $\Rightarrow$ (4). Let G be any l-preopen set in Y. Then from the definition of l-preopen sets and (3). it follows that
$f^{-1}(G)\subseteq f^{-1}\left(i_l\left(c_l(G)\right)\right)=X-f^{-1}\left(c_l(Y-c_l(G))\right)\subseteq X-c_m\left(f^{-1}(Y-c_l(G))\right)$
$=i_m\left(f^{-1}\left(c_l(G)\right)\right)$. Hence we have $f^{-1}(G)\subseteq i_m\left(f^{-1}\left(c_l(G)\right)\right)$.

(4) $\Rightarrow$ (1). Since every l-open set is l-preopen, from (4) and Theorem 2.3.6 (2), it follows f is weakly (m,l)-continuous.

2.4 Generalized Continuous Function Defined By Generalized Open Sets on Generalized Topological Spaces.

In this section we study generalized continuous functions defined by generalized open (=μ-a - open, μ-semi-open, μ-preopen, μ-b -

open) sets in generalized topological spaces which are generalized (m, l)-continuous functions. We investigate characterizations and relationships among such functions.

Definition 2.4.1. [W.K. Min, 2010]. Let (X, m) *and* (Y, l) be GTS's. Then a function $f : X \to Y$ is said to be

(1) (a, l)-continuous if for each l-open set U in Y, $f^{-1}(U)$ is m-a-open in X.

(2) (s, l)-continuous if for each l-open set U in Y, $f^{-1}(U)$ is m-semi-open in X.

(3) (p, l)-continuous if for each l-open set U in Y, $f^{-1}(U)$ is m-pre-open in X.

(4) (b, l)-continuous if for each l-open set U in Y, $f^{-1}(U)$ is m-b-open in X.

Remark 2.4.2. [W.K. Min, 2010]. Let $f : (X, m) \to (Y, l)$ be a function between GTS's (X, m) *and* (Y, l). Then we have the following implications (but the reverse relations may not be true in general):

$$\begin{array}{ccccc} & & \nearrow (s, l)\text{-continuous} \searrow & & \\ (m, l)\text{-continuous} & \rightarrow & (a, l)\text{-continuous} & \rightarrow & (b, l)\text{-continuous} \\ & & \searrow (p, l)\text{-continuous} \nearrow & & \end{array}$$

Example 2.4.3. Let $X = \{a,b,c,d\}$ and $m = \{f, \{a\}, \{a,b,c\}\}$ a GT on X. Consider a function $f : (X, m) \to (X, m)$. Define by $f(a) = f(b) = a$, $f(c) = b$, $f(d) = d$. Then $f^{-1}(\{a,b,c\}) = \{a,b,c\}$ and $f^{-1}(\{a\}) = \{a,b\}$. Now $\{a,b\}$ is m-α-open but not m-open. Hence f is (a, m)-continuous but not (m, m)-continuous.

Example 2.4.4. Let $X = \{a,b,c\}$ and $m = \{f, \{a,b\}\}$ a GT on X. Then $a = \{f, \{a,b\}\}$, $p = \{f, \{a\}, \{b\}, \{a,b\}\}$. Define a function $f : (X, m) \to (X, m)$ as follows: $f(a) = a$, $f(b) = f(c) = c$. Then f is (p, m)-continuous but it is not (a, m)-continuous.

Example 2.4.5. Let $X = \{a,b,c\}$ and $m = \{f, \{a\}, \{b\}, \{a,b\}\}$ a GT on X. Then $a = p = m$, $s = b = m \cup \{\{c\}, \{a,c\}, \{b,c\}, X\}$.

(1) Consider a function $f:(X,m)\to(X,m)$ defined by $f(a)=a,\ f(b)=f(c)=b$. Then f is (s,m)-continuous but it is not (a,m)-continuous.

(2) Consider a function $f:(X,m)\to(X,m)$ defined by $f(a)=a,\ f(b)=f(c)=a$. Then f is (b,m)-continuous without being (p,m)-continuous.

Theorem 2.4.6. [W.K. Min, 2010]. Let $f:(X,m)\to(Y,l)$ be a function between the GTS's (X,m) and (Y,l). Then the following statements are equivalent:

(1) f is (a,l)-continuous.

(2) For every l-closed subset F in Y, $f^{-1}(F)$ is m-a-closed in X.

(3) $c_m(i_m(c_m(f^{-1}(B))))\subseteq f^{-1}(c_l(B))$ for every subset B in Y .

(4) $f^{-1}(i_l(B))\subseteq i_m(c_m(i_m(f^{-1}((B)))))$ for every subset B in Y .

(5) $f(c_m(i_m(f^{-1}((A)))))\subseteq c_l(f(A))$ for every subset A in X.

Proof. (1) $\Rightarrow$ (2). Let F be l-closed then Y-F is l-open since f is (a,l)-continuous then $f^{-1}(Y-F)$ is μ-α-open then $X-f^{-1}(Y-F)$ is μ-α-closed then $f^{-1}(F)$ is μ-α-closed.

(2) $\Rightarrow$(1). Let U be a l-open subset of Y then Y-U is l-closed Suppose $f^{-1}(F)$ is μ-α-closed then $X-f^{-1}(Y-F)$ is μ-α-closed then $f^{-1}(Y-F)$ is μ-α-open then f is (a,l)-continuous.

(2) $\Rightarrow$(3). For $B\subseteq Y$, since $c_l(B)$ is a l-closed set in Y, by the hypothesis, $f^{-1}(c_l(B))$ is m-a-closed in X. So

$$f^{-1}(c_l(B))\supseteq c_m(i_m(f^{-1}(c_l(B))))\supseteq c_m(i_m(c_m(f^{-1}(c_l(B)))))$$
$$\supseteq c_m(i_m(c_m(f^{-1}(B)))).$$

Hence $c_m(i_m(c_m(f^{-1}(B))))\subseteq f^{-1}(c_l(B))$.

(3) $\Leftrightarrow$ (4). It follows from Theorem 1.2.3.

(3) $\Rightarrow$ (5). For any subset A in X, from (3), it follows

$$c_m(i_m(c_m(A))\subseteq c_m(i_m(c_m(f^{-1}(f(A)))))\subseteq f^{-1}(c_l(f(A)))$$

Hence we have $f(c_m(i_m(c_m(A))))\subseteq c_l(f(A))$.

(5) $\Rightarrow$ (2). Let F be any l-closed set in Y. By (5),

$$f(c_m(i_m(c_m(f^{-1}(F)))))\subseteq c_l(f(f^{-1}(F)))\subseteq c_l(F).$$

This implies $c_m(i_m(c_m(f^{-1}(F)))\subseteq f^{-1}(c_l(F))=f^{-1}(F)$. Thus $f^{-1}(F)$ is m-a-closed in X and hence f is (a,l)-continuous.

In the same manner, we have the following theorems.

Theorem 2.4.7. [W.K. Min, 2010]. Let $f:(X,m)\to(Y,l)$ be a function between the GTS's (X,m) and (Y,l). Then the following are equivalent:

(1) f is (s,l)-continuous.

(2) For every l-closed subset F in Y, $f^{-1}(F)$ is m-semi-closed in X.

(3) $i_m(c_m(f^{-1}(B)))\subseteq f^{-1}(c_l(B))$ for every subset B in Y.

(4) $f^{-1}(i_l(B))\subseteq c_m(i_m(f^{-1}(B)))$ for every subset B in Y.

(5) $f(i_m(c_m(A)))\subseteq c_l(f(A))$ for every subset A in X.

Proof. It is similar to the proof of Theorem 2.4.6.

Theorem 2.4.8. [W.K. Min, 2010]. Let $f:(X,m)\to(Y,l)$ be a function between the GTS's (X,m) and (Y,l). Then the following are equivalent:

(1) f is (p,l)-continuous.

(2) For every l-closed subset F in Y, $f^{-1}(F)$ is m-pre-closed in X.

(3) $c_m(i_m(f^{-1}(B)))\subseteq f^{-1}(c_l(B))$ for every subset B in Y.

(4) $f^{-1}(i_l(B))\subseteq i_m(c_m(f^{-1}((B)))$ for every subset B in Y.

(5) $f(c_m(i_m(A)))\subseteq c_l(f(A))$ for every subset A in X.

Proof. It is similar to the proof of Theorem 2.4.6.

Theorem 2.4.9. [W.K. Min, 2010]. Let $f:(X,m)\to(Y,l)$ be a function between the GTS's (X,m) and (Y,l). Then the following are equivalent:

(1) f is (b,l)-continuous.

(2) For every l-closed subset F in Y $f^{-1}(F)$ is m-b-closed in X.

(3) $i_m(c_m(i_m(f^{-1}(B))))\subseteq f^{-1}(c_l(B))$ for every subset B in Y.

(4) $f^{-1}(i_l(B))\subseteq c_m(i_m(c_m(f^{-1}(B))))$ for every subset B in Y.

(5) $f(i_m(c_m(i_m(A))))\subseteq c_l(f(A))$ for every subset A in X.

Proof. It is similar to the proof of Theorem 2.4.6.

Remark 2.4.10. [W.K. Min, 2010]. Let $f:(X,m)\to(Y,l)$ be a function between GTS's (X,m) and (Y,l). In the particular case (X,m) and (Y,l) are topological spaces, every (a,l)-continuous (resp., (s,l)-continuous, (p,l)-continuous, (b,l)-continuous) function is a-continuous [A. S. Mashhour, 1983], (resp., semicontinuous [N. Levine,1963], precontinuous [A. S. Mashour, 1982], b-continuous [V. Popa and T. Noiri,1992/1993]).

Let m be a GT on a nonempty set X and $S\subseteq X$. The m-a-closure (resp., m-semi-closure, m-preclosure, m-b-closure) of a subset

S of X, denoted by $c_a(S)$ (resp., $c_s(S)$, $c_p(S)$, $c_b(S)$), is the intersection of m-a-closed (resp., m-semi-closed, m-preclosed, m-b-closed) sets including S [A. Csaszar, 2005].

The m-a-interior (resp., m-semi-interior, m-preinterior, m-b-interior) of a subset S of X, denoted by $i_a(S)$ (resp., $i_s(S)$, $i_p(S)$, $i_b(S)$), is the union of m-a-open (resp., m-semi-open, m-preopen, m-b-open) sets contained in S [A. Csaszar,2005].

Lemma 2.4.11. [A. Csaszar, 2005]. Let m be a GT on a nonempty set X and $S \subseteq X$. Then

(1) $i_a(A) = A \cap i_m(c_m(i_m((A))))$, $c_a(A) = A \cup c_m(i_m(c_m(A)))$;

(2) $i_s(A) = A \cap c_m(i_m(A))$, $c_s(A) = A \cup i_m(c_m(A))$.

Lemma 2.4.12. [W.K. Min, 2010]. Let m be a GT on a nonempty set X and $A \subseteq X$. Then for $m = a, s, p, b$;

(1) $i_m(A) = X - c_m(X - A)$,

(2) $c_m(A) = X - i_m(X - A)$.

Proof. We only show $i_a(A) = X - c_s(X - A)$. For any $A \subseteq X$,

$X - c_a(X - A) = X - \cap\{F \subseteq X : X - A \subseteq F \text{ and } X - F \in a\}$

$= \cup\{X - F \subseteq X : X - F \subseteq A \text{ and } X - F \in a\} = i_a(A)$.

Theorem 2.4.13. [W.K. Min, 2010]. Let $f : X \to Y$ be a function between the GTS's (X, m) and (Y, l). Then the following statements are equivalent:

(1) f is (a, l)-continuous.

(2) $c_a(f^{-1}(B) \subseteq f^{-1}(c_l(B))$ for every subset B of Y.

(3) $f^{-1}(i_l(B)) \subseteq i_a(f^{-1}(B))$ for every subset B of Y.

(4) $f(c_a(A) \subseteq c_l(f(A))$ for every subset A of X.

Proof. (1) $\Rightarrow$ (2). For $B \subseteq Y$, from Theorem 2.4.6 and Lemma 2.4.11, it follows

$$f^{-1}(c_l(B)) = f^{-1}(c_l(B)) \cup f^{-1}(B)$$
$$\supseteq c_m(i_m(c_m(f^{-1}(B)))) \cup f^{-1}(B) = c_a(f^{-1}(B)).$$

Hence we have $c_a(f^{-1}(B)) \subseteq f^{-1}(c_l(B))$.

(2) $\Leftrightarrow$ (3). It follows from Lemma 2.4.12.

(2) $\Rightarrow$ (4). The proof is obvious.

(4) $\Rightarrow$ (1). Let F be any l-closed set in Y. Then $c_l(F) = F$, by (4) $f(c_a(f^{-1}(F))) \subseteq c_l(f(f^{-1}(F)) \subseteq c_l(F) = F$. This implies that

$c_a\left(f^{-1}(F)\right) \subseteq f^{-1}(F)$ and so $f^{-1}(F)$ is m-s-closed in X. Hence by Theorem 2.4.6, f is (a, l)-continuous.

Theorem 2.4.14. [W.K. Min, 2010]. Let $f: X \rightarrow Y$ be a function between the GTS's (X, m) and (Y, l). Then the following statements are equivalent:

(1) f is (s, l)-continuous.

(2) $c_s(f^{-1}(B) \subseteq f^{-1}(c_l(B))$ for every subset B of Y.

(3) $f^{-1}(i_l(B)) \subseteq i_s(f^{-1}(B))$ for every subset B of Y.

(4) $f(c_s(A) \subseteq c_l(f(A))$ for every subset A of X.

Proof. It is similar to the proof of Theorem 2.4.13.

Theorem 2.4.15. [W.K. Min, 2010]. Let $f: X \rightarrow Y$ be a function between the GTS's (X, m) and (Y, l). Then the following statements are equivalent:

(1) f is (p, l)-continuous.

(2) $c_p(f^{-1}(B) \subseteq f^{-1}(c_l(B))$ for every subset B of Y.

(3) $f^{-1}(i_l(B)) \subseteq i_p(f^{-1}(B))$ for every subset B of Y.

(4) $f(c_p(A) \subseteq c_l(f(A))$ for every subset A of X.

Proof. (1) $\Rightarrow$ (2). For $B \subseteq Y$, since $c_l(B)$ is l-closed in Y, by Theorem 2.4.8, $f^{-1}(c_l(B)))$ is l-preclosed in Y. This implies $c_p\left(f^{-1}(B)\right) \subseteq c_p\left(f^{-1}\left(c_l(B)\right)\right) = f^{-1}\left(c_l(B)\right)$ So $c_p(f^{-1}(B)) \subseteq f^{-1}(c_l(B))$.

(2) $\Rightarrow$ (1). Let F be any l-closed set in Y. Then from (2), $c_p(f^{-1}(F)) \subseteq f^{-1}(c_l(F)) = f^{-1}(F)$. So $f^{-1}(F)$ is m-pre-closed and hence by Theorem 2.4.8 (2), f is (p, l)-continuous.

(2) $\Leftrightarrow$ (3). It follows from Lemma 2.4.12.

(2) $\Leftrightarrow$ (4). The proof is obvious.

Theorem 2.4.16. [W.K. Min, 2010]. Let $f: X \rightarrow Y$ be a function on the GTS's (X, m) and (Y, l). Then the following statements are equivalent:

(1) f is (b, l)-continuous.

(2) $c_b(f^{-1}(B) \subseteq f^{-1}(c_l(B))$ for every subset B of Y.

(3) $f^{-1}(i_l(B)) \subseteq i_b(f^{-1}(B))$ for every subset B of Y.

(4) $f(c_b(A) \subseteq c_l(f(A))$ for every subset A of X.

Proof. It is similar to Theorem 2.4.15.

2.5 Contra (m, l)-Continuous Functions.

In this section we study a notion called contra (m, l)-continuous functions as functions on generalized topological spaces

[A. Csaszar , 2002]. Some characterizations and several properties of such functions are studied in this section.

Definition 2.5.1. [A. AL-Omari, T. Noiri, 2012]. Let $f:(X,m)\rightarrow(Y,l)$ be a function on generalized topological spaces. Then the function f is said to be contra (m,l)-continuous if $f^{-1}(V)$ is m-closed in X for each l-open set V of Y.

Definition 2.5.2. [B. Roy,2010]. Let (X,m) be a GTS. The generalized μ-kernel of $A\subseteq X$ is denoted by m-ker(A) and defined as m-ker(A) = **I** $\{G\in m$: A⊆G}.

Example 2.5.3. Let $X=\{a,b,c,d\}$ Consider the generalized topology $m=\{f,\{a,c\},\{b,c\},\{a,b,c\}\}$. on X. Let $A=\{a\}$ then m-ker$(A)=\{a,c\}$. Since $\{a,c\}\cap\{a,b,c\}=\{a,c\}$.

Lemma 2.5.4. [B. Roy,2010]. Let (X,m) be a GTS and $A\subseteq X$, then

$$m\text{-ker}(A)=\{x\in X\ :\ c_m(\{x\})\cap A\neq f\}.$$

Theorem 2.5.5. [A. AL-Omari, T. Noiri, 2012]. For a function $f:(X,m)\rightarrow(Y,l)$, the following properties are equivalent :

(1) f is contra (m,l)-continuous ;

(2) for every λ-closed subset F of Y, $f^{-1}(F)$ is μ-open in X;

(3) for each $x\in X$ and each λ-closed subset F of Y containing $f(x)$, there exists a μ-open set U of X containing x such that $f(U)\subseteq F$;

(4) $f\ (c_m((A))\subseteq l-Ker(f(A))$ for every subset A of X;

(5) $(c_m(f^{-1}(B))\subseteq f^{-1}(l-Ker(B))$ for every subset B of Y.

Proof. The equivalence of (1) and (2) and the implication (2) ⇒ (3) are obvious.

(3) ⇒ (2). Let F be any λ-closed set of Y and $x\in f^{-1}(F)$. Then $f(x)\in F$ and there exists a m-open set U_x containing x such that $f(\{U_x\})\subseteq F$. Therefore, we obtain $f^{-1}(F)=\cup\{U_x\mid x\in f^{-1}(F)\}$ and $f^{-1}(F)$ is m-open.

(2) ⇒ (4). Let A be any subset of X. Suppose that $y\notin l\text{-}Ker(f(A))$. Then by Lemma 2.5.4, there exists a λ-closed set F containing y such that $f(A)\cap F=f$. Thus, we have $A\cap f^{-1}(F)=f$ and since $f^{-1}(F)$ is μ-open then we have $c_m(A)\cap f^{-1}(F)=f$. Therefore we obtain

$f\left(c_m(A)\right)\cap F=f$ and $y\notin f(c_m(A))$. This implies that $f(c_m(A))\subseteq l\text{-}Ker\left(f(A)\right)$.

(4) ⇒ (5). Let B be any subset of Y. By (4), we have $f\left((c_m(f^{-1}(B))\right)\subseteq l\text{-}Ker\left(f(f^{-1}(B)\right)\subseteq l\text{-}Ker(B)$ and thus $c_m(f^{-1}(B))\subseteq f^{-1}\left(l\text{-}Ker(B)\right)$.

(5) ⇒ (1). Let V be any λ-open set of Y. Then, we have $c_m(f^{-1}(V))\subseteq f^{-1}\left(l\text{-}Ker(V)\right)=f^{-1}(V)$ and $c_m(f^{-1}(V))=f^{-1}(V)$. This shows that $f^{-1}(V)$ is m-closed in X.

Let m be a GT on X. A mapping $k:m\rightarrow \exp X$ is called an enlargement [A. Csaszar, 2008] on X if $M\subseteq kM\ (=k(M))$ whenever $M\in m$. $kM=c_m(M)$ and $kM=i_m\left(c_m(M)\right)$ are examples for an enlargement κ.

Definition 2.5.6. [W.K. Min and Y. K. Kim, 2010]. Let m be a GT on a non-empty set X and let $k:m\rightarrow \exp X$ be an enlargement for m. (X,m) is *E*-regular if and only if for each $x\in X$ and each m-open set U containing x, there exists a m-open set V such that $x\in V\subseteq k(V)\subseteq U$.

Theorem 2.5.7. [A. AL-Omari, T. Noiri, 2012]. If a function $f:(X,m)\rightarrow(Y,l)$ is contra (m,l)-continuous and Y is *E*-regular, where $k=c_l$, then f is (m,l)- continuous**.**

Proof. Let x be an arbitrary point of X and V be a λ-open set of Y containing $f(x)$. Since Y is *E*-regular, there exists a λ-open set W in Y containing $f(x)$ such that $c_l(W)\subseteq V$. Since f is contra (m,l)-continuous, so by Theorem 2.5.5. (3) there exists a μ-open set U containing x such that $f(U)\subseteq c_l(W)$. Then $f(U)\subseteq c_l(W)\subseteq V$. Hence, f is (m,l)-continuous.

Definition 2.5.8. [A. AL-Omari, T. Noiri, 2012]. A function $f:(X,m)\rightarrow(Y,l)$ is said to satisfy the (m,l)-interiority condition if $i_m\left(f^{-1}(c_l(V))\right)\subseteq f^{-1}(V)$ for each λ-open set V of (Y,l).

Theorem 2.5.9. [A. AL-Omari, T. Noiri, 2012]. If $f:(X,m)\rightarrow(Y,l)$ is a contra (m,l)-continuous function and satisfies the (m,l)-interiority condition, then f is (m,l)-continuous.

Proof. Let V be any λ-open set of Y. Since f is contra (m,l)-continuous and $c_l(V)$ is λ-closed, by Theorem 2.5.5. $f^{-1}(c_l(V))$ is μ-open in X. By the hypothesis on f, $f^{-1}(V)\subseteq f^{-1}(c_m(V))$ $\subseteq i_m\left(f^{-1}(c_m(V))\right)\subseteq i_m\left(f^{-1}(V)\right)\subseteq f^{-1}(V)$. Therefore, we obtain

$f^{-1}(V)=i_m\left(f^{-1}(V)\right)$ and consequently $f^{-1}(V)$ is μ-open. This shows that f is a (m,l)-continuous function.

Definition 2.5.10. [A. AL-Omari, T. Noiri, 2012]. Let $f:(X,m)\to(Y,l)$ be a function on generalized topological spaces. Then the function f is said to be almost (m,l)-open if $f(U)\subseteq i_l\left(c_l\left(f(U)\right)\right)$ for every $U\in m$.

Theorem 2.5.11. [A. AL-Omari, T. Noiri, 2012]. If $f:(X,m)\to(Y,l)$ is an almost (m,l)-open and contra (m,l)-continuous function, then f is almost (m,l)-continuous.

Proof. Let $x\in X$ and V be any λ-open set of Y containing $f(x)$. Then $c_l(V)$ is a λ-closed set containing $f(x)$. Since f is contra (m,l)-continuous, by Theorem 2.5.5. There exists $U\in m$ containing x such that $f(U)\subseteq c_l(V)$. Since f is almost (m,l)-open, $f(U)\subseteq i_l\left(c_l(f(U)\right)\subseteq i_l\left(c_l(V)\right)$. Hence f is almost (m,l)-continuous.

Definition 2.5.12. [A. AL-Omari, T. Noiri, 2012]. A GTS (Y,l) is said to be almost-λ-regular if for any λ-regular-closed set F of Y, that is, $F=c_l\left(i_l(F)\right)$ and any $y\notin F$, there exist disjoint λ-open sets U and V such that $y\in U$ and $F\subseteq V$.

Theorem 2.5.13. [A. AL-Omari, T. Noiri, 2012]. If $f:(X,m)\to(Y,l)$ is contra (m,l)-continuous and (Y,l) is almost-λ-regular, then f is almost (m,l)-continuous.

Proof. Let $x\in X$ and V be any λ-open set of Y containing $f(x)$. Since (Y,l) is almost-λ-regular, we can claim that there exists a λ- regular open set G of Y such that $f(x)\in G\subseteq c_l G\subseteq i_l\left(c_l(V)\right)$. Since V is λ-open, then $F=c_l\left(i_l(Y-V)\right)$ is λ-regular closed and $f(x)\notin F$. Therefore, there exist λ-open sets U_1 and U_2 such that $f(x)\in U_1$, $F\subseteq U_2$ and $U_1\cap U_2=f$. Then $c_l(U_1)\cap U_2=f$ and hence $c_l(U_1)\subseteq Y-U_2\subseteq Y-F$. Thus $f(x)\in U_1\subseteq c_l(U_1)\subseteq Y-F$. Again $U_1\subseteq i_l\left(c_l(U_1)\right)\subseteq c_l(U_1)\subseteq Y-F$. Therefore, if $G=i_l\left(c_l(U_1)\right)$, then $U_1\subseteq G\subseteq c_l(G)\subseteq c_l(U_1)\subseteq i_l\left(c_l(V)\right)$. Hence $f(x)\in G\subseteq c_l(G)\subseteq i_l\left(c_l(V)\right)$. Since f is contra (m,l)-continuous and $c_l(G)$ is λ-closed in Y, by Theorem 2.5.5. There exists $U\in m$ containing x such that $f(U)\subseteq c_l(G)\subseteq i_l\left(c_l(V)\right)$. Hence f is almost (m,l)- continuous.

Definition 2.5.14. [S. Willard, 1970] . A topological space (Y,s) is said to be extremally disconnected (briefly E.D.) if the closure of each open set of Y is open in Y.

Example 2.5.15. Let $X=\{a,b,c\}$. Consider the generalized topologies $m=\{f,X,\{a,b\},\{c\}\}$. Now the closure of each open set of μ is open in μ. Hence (X,t) is extremally disconnected.

Theorem 2.5.16. [A. AL-Omari, T. Noiri, 2012]. If $f:(X,m)\to(Y,s)$ is contra (m,s)-continuous and (Y,s) is E.D., then f is almost (m,s)-continuous.

Proof. Let $x\in X$ and V be an open set of Y containing $f(x)$. Since (Y,s) is E.D., $cl(V)$ is clopen. Since f is contra (m,s)-continuous, by Theorem 2.5.5. There exists $U\in m$ containing x such that $f(U)\subseteq cl(V)=(c_l(V))$. Hence, f is almost (m,s)-continuous.

We will denote by M_m the union of all μ-open sets in a GTS (X,m).

Definition 2.5.17. [A. AL-Omari, T. Noiri, 2012]. A GTS (X,m) is said to be μ-connected if M_m cannot be expressed as the union of two disjoint non-empty μ-open subsets of X.

Example 2.5.18. (1) Let $X=\{a,b,c\}$ consider the generalized topologies $m=\{f,\{a,b\},\{c\},\{a,b,c\}\}$. Now $M_m=\{a,b,c\}=\{a,b\}\cup\{c\}$. Then (X,m) is not μ-connected.

(2) Let $X=\{a,b,c\}$ consider the generalized topologies $m=\{f,\{a,b\},\{b,c\},\{a,b,c\}\}$. Now then (X,m) is μ-connected.

Theorem 2.5.19. [A. AL-Omari, T. Noiri, 2012]. Let $f:(X,m)\to(Y,l)$ be a (m,l)-continuous surjection and $f(M_m)\subseteq M_l$. If (X,m) is μ-connected, then (Y,l) is λ-connected.

Proof. Let $f:(X,m)\to(Y,l)$ be a (m,l)-continuous function of a μ- connected space (X,m) onto (Y,l). If possible, let Y be λ-disconnected. Let A and B form a disconnection of M_l. Then A and B are λ-open and $M_l=A\cup B$ where $A\cap B=f$. Since f is a (m,l)-continuous function, $M_m=f^{-1}(M_l)=f^{-1}(A\cup B)$ $=f^{-1}(A)\cup f^{-1}(B)$, where $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty μ-open sets in X. Also $f^{-1}(A)\cap f^{-1}(B)=f$. Hence X is not μ-connected. This is a contradiction. Therefore Y is λ-connected.

Proposition 2.5.20. [A. AL-Omari, T. Noiri, 2012]. Let $f:(X,m)\to(Y,l)$ be a (m,l)-continuous surjection. If Y is λ-open and (X,m) is μ-connected, then (Y,l) is λ- connected.

Definition 2.5.21. [M. S. Sarsak, 2011]. A GTS (X, m) is said to be m-T_2 if for each pair of distinct points x and y in X, there exist disjoint μ-open sets U and V containing x and y, respectively.

Definition 2.5.22. [A. AL-Omari, T. Noiri, 2012]. A GTS (X, m) is said to be μ-Urysohn if for each pair of distinct points x and y in X, there exist μ-open sets U and V such that $x \in U$, $y \in V$ and $c_m(U) \cap c_m(V) = f$.

Theorem 2.5.23. [A. AL-Omari, T. Noiri, 2012]. Let (X, m) be a GTS. If the following three assumptions are satisfied :

(1) for each pair of distinct points x and y in X there exists a function f of (X, m) into a GTS (Y, l) such that $f(x) \neq f(y)$,

(2) (Y, l) is a λ-Urysohn space,

(3) f is contra (m, l)-continuous at x and y, then (X, m) is m-T_2.

Proof. Let x and y be any distinct points in X. Then, there exist a λ-Urysohn space (Y, l) and a function $f:(X, m) \to (Y, l)$ such that $f(x) \neq f(y)$ and f is contra (m, l)-continuous at x and y. Let $a = f(x)$ and $b = f(y)$. Then $a \neq b$. Since Y is a λ-Urysohn space, there exist λ-open sets V and W containing a and b, respectively, such that $c_l(V) \cap c_l(W) = f$. Since f is contra (m, l)-continuous at x and y, then there exist μ-open sets A and B containing a and b, respectively, such that

$f(A) \subseteq c_l(V)$ and $f(B) \subseteq c_l(W)$. Then $f(A) \cap f(B) \neq f$, so

$A \cap B = f$. Hence X is m-T_2.

Corollary 2.5.24. [A. AL-Omari, T. Noiri, 2012]. Let $f:(X, m) \to (Y, l)$ be a contra (m, l)-continuous injection. If Y is a λ-Urysohn space, then X is m-T_2.

Theorem 2.5.25. [A. AL-Omari, T. Noiri, 2012]. If a function $f:(X, m) \to (Y, l)$ is contra (m, l)- continuous, then f is weakly (m, l)-continuous.

Proof. Let $x \in X$ and V be any λ-open set of Y containing *f(x).* Since $c_l(V)$ is λ-closed in Y, by Theorem 2.5.5. (2), $f^{-1}(c_l(V))$ is μ-open in X. Set $U = f^{-1}(c_l(V))$ then we have $x \in U \in m$ and $f(U) \subseteq c_l(V)$. This shows that f is weakly (m, l)-continuous.

Example 2.5.26. Let $X = \{a, b, c\}$, $m = \{f, X, \{a, b\}, \{a, c\}\}$ and $l = \{f, \{a\}, \{b\}, \{a, b\}\}$. Let $f:(X, m) \to (Y, l)$ be a function defined by $f(a) = c$, $f(b) = b$ and $f(c) = a$. Then f is weakly (m, l)-continuous but not contra-(m, l)-continuous.

Theorem 2.5.27. [A. AL-Omari, T. Noiri, 2012]. $f, g:(X, m) \to (Y, l)$ are contra (m, l)- continuous, µ is a topology and Y is λ-Urysohn, then $E = \{x \in X : f(x) = g(x)\}$ is µ-closed in *X*.

Proof. Let $x \in X - E$. Then $f(x) \neq g(x)$. Since Y is λ-Urysohn, there exist λ-open sets V and W such that $f(x) \in V$, $g(x) \in W$ and $c_l(V) \cap c_l(W) = f$. Since f and g are contra (m, l)-continuous, then $f^{-1}(c_l(V))$ and $g^{-1}(c_l(W))$ are µ-open sets in *X*. Let $U = f^{-1}(c_l(V))$ and $G = g^{-1}(c_l(W))$. Then U and V are µ-open sets containing *x*. Set $A = U \cap G$, thus A is µ-open in *X*. Hence $f(A) \cap g(A) = f(U \cap G) \cap g(U \cap G) \subseteq f(U) \cap g(G) = c_l(V) \cap c_l(W) = f$. Therefore $A \cap E = f$ and $x \notin c_m(E)$. Hence E is µ-closed in X.

A subset A of a generalized topological space (X, m) is said to be µ-dense in *X*. If $c_l(A) = X$.

Theorem 2.5.28. [A. AL-Omari, T. Noiri, 2012]. Let $f, g:(X, m) \to (Y, l)$ be functions and µ be a topology. If Y is λ-Urysohn, f and g are contra (m, l)-continuous and $f = g$ on a µ-dense set $A \subseteq X$, then $f = g$ on X.

Proof. Since f and g are contra (m, l)-continuous and Y is λ-Urysohn, by the previous theorem $E = \{x \in X : f(x) = g(x)\}$ is µ-closed in *X*. By assumption, we have $f = g$ on a µ-dense set $A \subseteq X$. Since $A \subseteq E$ and A is a µ-dense set in *X*, then $X = c_m(A) \subseteq c_m(E) = E$. Hence $f = g$ on X.

Definition 2.5.29. [A. AL-Omari, T. Noiri, 2012]. A function $f:(X, m) \to (Y, l)$ is said to be

(1) contra (a, l)-continuous if $f^{-1}(V)$ is m-a -closed in X for each λ-open set V of Y,

(2) contra (s, l)-continuous if $f^{-1}(V)$ is *µ*-semi-closed in X for each λ- open set V of Y,

(3) contra (p, l)-continuous if $f^{-1}(V)$ is *µ*-preclosed in X for each λ- open set V of Y,

(4) contra (b, l)-continuous if $f^{-1}(V)$ is *µ*-*β*-closed in X for each λ-open set V of Y.

Remark 2.5.30. [A. AL-Omari, T. Noiri, 2012]. Let $f:(X, m) \to (Y, l)$ be a function. Then the following implications is true but the reverse relations are not true in general:

contra-(m,l)-continuous à contra-(a,l)-continuous à contra-(s,l)-continuous

↓ ↓

contra-(p,l)-continuous à contra-(b,l)-continuous

Example 2.5.31. Let $X=\{a,b,c\}$ and $l=\{f,\{a\},\{b\},\{a,b\}\}$ a GT on X. Then $a(l)=p(l)=l$ and $s(l)=b(l)=l\cup\{\{c\},\{a,c\},\{b,c\},X\}$.
Let $f:(X,m)\rightarrow(Y,l)$ be the identity function. Then f is contra(s,l)- continuous but not contra (p,l)-continuous.

Example 2.5.32. Let $X=\{a,b,c\},m=\{f,X,\{a,b\},\{a,c\}\}$ and $l=\{f,\{a\},\{b\},\{a,b\}\}$. Then $p(m)=m\cup\{\{a\},\{b,c\}\}$ and $s(m)=m$. Let $f:(X,m)\rightarrow(Y,l)$ be a function defined by $f(a)=c$, $f(b)=b$ and $f(c)=a$. Then f is contra (p,l)-continuous but not contra (s,l)-continuous.

Example 2.5.33. Let $X=\{a,b,c,d\}$ and $m=\{f,X,\{a\},\{a,b,c\}\}$ a GT on X. Then $a(m)=m\cup\{\{a,b\},\{a,c\},\{a,d\},\{a,b,d\},\{a,c,d\}\}$. Let $f:(X,m)\rightarrow(Y,l)$ be a function defined by $f(a)=d$, $f(b)=a$, $f(c)=b$ and $f(d)=c$. Then f is contra (a,m)-continuous but not contra (m,l)- continuous.

Theorem 2.5.34. Let $f:X\rightarrow Y$ be a function from two GTS's (X,m) and (Y,l). Then the following conditions are equivalent:

(1) f is contra (p,m)-continuous,

(2) $f^{-1}(A)$ is μ-preopen in X for every l-closed set A in Y,

(3) $f^{-1}(A)\subseteq i_m\left(c_m\left(f^{-1}\left(c_l(A)\right)\right)\right)$ for every subset A in Y,

(4) $c_m\left(i_m\left(f^{-1}\left(i_l(A)\right)\right)\right)\subseteq f^{-1}(A)$ for every subset A in Y,

(5) $A\subseteq i_m\left(c_m\left(f^{-1}\left(c_l\left(f(A)\right)\right)\right)\right)$ for every subset A in Y.

Proof . (1)⇔(2) The proof is obvious.

(2) ⇒(3) let $A \subseteq Y$ .Then $c_l(A)$ is a l -closed set in Y. by (2) implies that

$f^{-1}(c_l(A))$ is a μ-preopen set in X. Therefore
$f^{-1}(c_l(A)) \subseteq i_m(c_m(f^{-1}(c_l(A))))$.
Hence $f^{-1}(A) \subseteq f^{-1}(c_l(A)) \subseteq i_m(c_m(f^{-1}(c_l(A))))$.

(3)⟺(4) The proof is obvious.

(3) ⇒(5) Let $A \subseteq X$. Then $f(A) \subseteq Y$ .by (3) implies that

$f^{-1}(f(A)) \subseteq i_m(c_m(f^{-1}(c_l(f(A)))))$.Therefore

$$A \subseteq f^{-1}(f(A)) \subseteq i_m(c_m(f^{-1}(c_l(f(A))))).$$

(5) ⟹ (2) Let A be l -closed in Y. Then $f^{-1}(A) \subseteq X$. By hypothesis

$$f^{-1}(A) \subseteq i_m\left(c_m(f^{-1}(c_l(f(f^{-1}(A)))))\right)$$
$$= i_m(c_m(f^{-1}(A))).$$

Hence $f^{-1}(A)$. Is μ-preopen in X.

Theorem 2.5.35. Let $f : X \to Y$ be a function from two GTS's (X, m) and (Y, l). Then the following conditions are equivalent:

(1) f is contra (s, m)-continuous,

(2) $f^{-1}(A)$ is μ-semi-open in X for every l -closed set A in Y,

(3) $f^{-1}(A) \subseteq c_m(i_m(f^{-1}(c_l(A))))$ for every subset A in Y,

(4) $i_m(c_m(f^{-1}(i_l(A)))) \subseteq f^{-1}(A)$ for every subset A in Y,

(5) $A \subseteq c_m\left(i_m\left(f^{-1}\left(c_l\left(f\left(A\right)\right)\right)\right)\right)$ for every subset A in Y.

Proof. **(1) ⇔ (2).** The proof is obvious.

(2)⟹(3). Let $A \subseteq Y$. Then $c_l\left(A\right)$ is a l-closed set in Y. By (2)

implies that $f^{-1}\left(c_l\left(A\right)\right)$ is a μ-semi-open set in X. Therefore

$f^{-1}\left(c_l\left(A\right)\right) \subseteq c_m\left(i_m\left(f^{-1}\left(c_l\left(A\right)\right)\right)\right)$. Hence
$f^{-1}\left(A\right) \subseteq f^{-1}\left(c_l\left(A\right)\right) \subseteq c_m\left(i_m\left(f^{-1}\left(c_l\left(A\right)\right)\right)\right)$.

(3)⟺(4). Can be proved by taking complement.

(3)⟹(5). Let A be a subset of X. Then $f\left(A\right) \subseteq Y$. By (3) implies

that $f^{-1}\left(f\left(A\right)\right) \subseteq c_m\left(i_m\left(f^{-1}\left(c_l\left(f\left(A\right)\right)\right)\right)\right)$. That is
$A \subseteq f^{-1}\left(f\left(A\right)\right) \subseteq c_m\left(i_m\left(f^{-1}\left(c_l\left(f\left(A\right)\right)\right)\right)\right)$.

(5)⟹(2). Let A be l -closed in Y. Then $f^{-1}(A) \subseteq X$. By (5) implies

that $f^{-1}(A) \subseteq c_m\left(i_m\left(f^{-1}\left(c_l\left(f\left(f^{-1}(A)\right)\right)\right)\right)\right) \subseteq c_m\left(i_m\left(f^{-1}\left(c_l(A)\right)\right)\right)$

$= c_m\left(i_m\left(f^{-1}(A)\right)\right)$. Hence $f^{-1}(A)$ is μ-semi-open in *X*.

CHABTER THREE
Some Mapping via Generalized Topological Spaces

3.1 Weakly (m, l)-Open Functions:

In section 3.1 and 3.2 we introduce weakly (m, l)-open functions and weakly (m, l)-closed functions and investigate charactrizations and relationships among such functions.

Definition 3.1.1. A functions $f:(X, m)\to(Y, l)$ is said to be (m, l)-open if for each μ-open set U of X, $f(U)$ is l-open.

Example 3.1.2. Let $X=\{a,b,c,d\}, m=\{f,\{b\},\{a,b\},\{b,c\},\{b,c,d\},X\}$ and $f:X\to X$ be a function defined by $f(a)=c$, $f(b)=b$, $f(c)=a$ and $f(d)=d$. Then f is a (m,m)-open function

Definition 3.1.3. A functions $f:(X, m)\to(Y, l)$ is said to be weakly (m, l)-open if $f(U)\subseteq i_l(f(c_m(U)))$ for each μ-open set U of X. Every (m, l)-open function is also weakly (m, l)-open, but the converse is not generally true.

Example 3.1.4. Let $X=\{a,b,c\}$ and $m=\{f,\{a\},\{b\},\{a,b\},\{a,c\},X\}$. Let $f:(X, m)\to(X, m)$ be a function defined by $f(a)=c$, $f(b)=b$ and $f(c)=a$. Then f is a weakly (m, l)-open function which is not (m, l)-open, since for $U=\{a\}$, $f(U)=\{c\}$ is not μ-open in X. f is weakly (m, l)-open since each μ-open satisfying $f(U)\subseteq i_l(f(c_m(U)))$.

Definition 3.1.5. the space (X, m) is μ-regular if and only if for each $x\in X$ and each μ-open set U containing x, there exists a μ-open set V such that $x\in V\subseteq c_m(V)\subseteq U$.

Theorem 3.1.6. Let (X, m) be a μ-regular space and a function $f:(X, m)\to(Y, l)$, the following conditions are equivalent:

(1) f is weakly (m, l)-open;

(2) $f(i_m(A))\subseteq i_l(f(A))$ for every subset A of X;

(3) $i_m(f^{-1}(B))\subseteq f^{-1}(i_l(B))$ for every subset B of Y;

(4) $f^{-1}(c_l(B))\subseteq c_m(f^{-1}(B))$ for every subset B of Y.

Proof. (1) ⇒ (2). Let A be any subset of X and $x \in i_m(A)$. Then, there exists a μ-open set U such that $x \in U \subseteq i_m(A) \subseteq A.$ Then, $f(x) \in f(U) \subseteq f(c_m(U)) \subseteq f(A).$ Since f is weakly (m, l)-open, $f(U) \subseteq i_l(f(c_m(U))) \subseteq i_l(f(A))$. This implies that $f(x) \in i_l(f(A))$. This shows that $x \in f^{-1}(i_l(f(A)))$. Thus, $i_m(A) \subseteq f^{-1}(i_l(f(A)))$, and so, $f(i_m(A)) \subseteq i_l(f(A))$.

(2)⇒ (3). Let B be any subset of Y. Then by (2), $f(i_m(f^{-1}(B))) \subseteq i_l(f(f^{-1}(B))) \subseteq i_l(B)$. Therefore, $i_m(f^{-1}(B)) \subseteq f^{-1}(i_l(B))$.

(3)⇒ (4). Let B be any subset of Y. Using (3), we have

$$X - c_m(f^{-1}(B)) = i_m(X - f^{-1}(B))$$
$$= i_m(f^{-1}(Y - B)) \subseteq f^{-1}(i_l(Y - B))$$
$$= f^{-1}(Y - c_l(B))$$
$$= X - f^{-1}(c_l(B)).$$

Therefore, we obtain $f^{-1}(c_l(B)) \subseteq c_m(f^{-1}(B))$.

(4) ⇒ (1). Let V be any μ-open set of X and $B = Y - f(c_m(V))$. By (4), $f^{-1}(c_l(Y - f(c_m(V)))) \subseteq c_m(f^{-1}(Y - f(c_m(V))))$. Therefore, we obtain $f^{-1}(Y - i_l(f(c_m(V)))) \subseteq c_m(X - f^{-1}(f(c_m(V))))$ $\subseteq c_m(X - c_m(V))$. Hence $V \subseteq i_m(c(V)) \subseteq f^{-1}(i_l(f(c_m(V))))$ and $f(V) \subseteq i_l(f(c_m(V)))$. This shows that f is weakly (m, l)-open.

Theorem 3.1.7. For a function $f:(X, m) \to (Y, l)$, the following conditions are equivalent:

(1) f is weakly (m, l)-open;

(2) For each $x \in X$ and each μ-open subset U of X containing x, there exists a l-open set V containing f(x) such that $V \subseteq f(c_m(U))$.

Proof. (1) ⇒ (2). Let $x \in X$ and U be an μ-open set in X with $x \in U$. Since f is weakly (m, l)-open, $f(x) \in f(U) \subseteq i_l(f(c_m(U)))$. Let $V = i_l(f(c_m(U)))$. Then V is μ-open and $f(x) \in V \subseteq f(c_m(U))$.

(2) ⇒ (1). Let U be a μ-open set in X and let $y \in f(U)$. It follows from (2) that $V \subseteq f(c_m(U))$ for some l-open set V in Y containing y. Hence, we have $y \in V \subseteq i_l(f(c_m(U)))$. This shows that $f(U) \subseteq i_l(f(c_m(U)))$. Thus f is weakly (m, l)-open.

Theorem 3.1.8. For a bijective function $f:(X, m) \to (Y, l)$, the following conditions are equivalent:

(1) f is weakly (m, l)-open;

(2) $c_l(f(i_m(F))) \subseteq f(F)$ for each μ-closed set F in X;

(3) $c_l(f(U)) \subseteq f(c_m(U))$ for each μ-open set U in X.

Proof. (1) ⇒ (2). Let F be a μ-closed set in X. Then we have $f(X - F) = Y - f(F) \subseteq i_l(f(c_m(X - F)))$, and so $Y - f(F) \subseteq Y - c_l(f(i_m(F)))$. Hence $c_l(f(i_m(F))) \subseteq f(F)$.

(2) ⇒ (3). Let U be an μ-open set in X. Since $c_m(U)$ is a μ-closed set and $U \subseteq i_m(c_m(U))$, by (2) we have $c_l(f(U)) \subseteq c_l(f(i_m(c_m(U))))$ $\subseteq f(c_m(U))$.

(3) ⇒ (1). Let V be a μ-open set of X. Then, we have

$Y - i_l(f(c_m(V))) = c_l(Y - f(c_m(V))) = c_l(f(X - c_m(V)))$

$\subseteq f(c_m(X - c_m(V))) = f(X - i(c_m(V))) \subseteq f(X - V) = Y - f(V)$.

Therefore, we have $f(V) \subseteq i_l(f(c_m(V)))$ and hence f is weakly (m, l)-open.

Theorem 3.1.9. Let X be a μ-regular space. A function $f:(X, m) \to (Y, l)$ is weakly (m, l)-open if and only if f is (m, l)-open.

Proof. The sufficiency is clear. For the necessity, let W be a nonempty open subset of X. For each x in W, let U_x be an μ-open set such that $x \in U_x \subseteq c_m(U_x) \subseteq W$. Hence we obtain that $W = \cup\{U_x : x \in W\} = \cup\{c_m(U_x) : x \in W\}$, and

$f(W) = \cup\{f(U_x) : x \in W\} \subseteq \cup\{i_l(f(c_m(Ux))) : x \in W\}$

$\subseteq i_l(f(\cup\{c_m(U_x) : x \in W\})) = i_l(f(W))$. Thus f is (m, l)-open.

Definition 3.1.10. A function $f:(X,m)\to(Y,l)$ is said to be strongly (m,l)-continuous if for every subset A of X, $f(c_m(A))\subseteq f(A)$.

Theorem 3.1.11. If $f:(X,m)\to(Y,l)$ is weakly (m,l)-open and strongly (m,l)-continuous, then f is (m,l)-open.

Proof. Let U be a μ-open subset of X. Since f is weakly (m,l)-open $f(U)\subseteq i_l(f(c_m(U)))$. However, because f is strongly (m,l)-continuous, $f(U)\subseteq i_l(f(U))$. *f(U)* is l -open. Therefore f is (m,l)-open.

Definition 3.1.12. A function $f:(X,m)\to(Y,l)$ is said to be contra (m,l)-closed if f(U) is a l -open set of Y, for each μ-closed set U in X.

Theorem 3.1.13. If $f:(X,m)\to(Y,l)$ is a contra (m,l)-closed function, then f is weakly (m,l)-open.

Proof. Let U be a μ-open subset of X. Then, we have $f(U)\subseteq f(c_m(U))=i_l(f(c_m(U)))$.

Example 3.1.14. The converse of Theorem 3.1.13, dose not hold.

Example 3.1.2. Shows that a weakly μ-open function need not be contra μ-closed, since $f(\{b,c\})=\{a,b\}$ is not μ-open in X.

Definition 3.1.15. A space X is said to be μ-hyperconnected if every non-empty μ-open subset of X is μ-dense in X.

Theorem 3.1.16. If X is a hyperconnected space, then a function $f:(X,m)\to(Y,l)$ is weakly (m,l)-open if and only if f(X) is μ-open in Y .

Proof. The sufficiency is clear. For the necessity observe that for any μ-open subset U of X, $f(U)\subseteq f(X)=i_l(f(X))=i_l(f(c_m(U)))$.

Hence f is weakly (m,l)-open.

Theorem 3.1.17. If $f:(X,m)\to(Y,l)$ is a weakly (m,l)-open bijective function of a space X onto a μ-connected space Y , then X is μ-connected.

Proof. Suppose that X is not μ-connected. Then there exist nonempty μ-open sets U and V such that U ∩ V = Ø and $U\cup V=X$. Hence we have f(U) ∩ f(V) = Ø and f(U) ∪ f(V) = Y . Since f is weakly (m,l)-open, we have $f(U)\subseteq i_l(f(c_m(U)))$ and $f(V)\subseteq i_l(f(c_m(U)))$.

Moreover U, V are μ-open and also μ-closed. We have $f(U)=i_l(f(U))$ and $f(V)=i_l(f(V))$. Hence, f(U) and f(V) are μ-open in Y. Thus, Y has been decomposed into two nonempty disjoint μ-open sets. This is contrary to the hypothesis that Y is a μ-connected space. Thus, X is μ-connected.

In addition to 3.1.6 We have more characterizations

Theorem 3.1.18. Let X be a μ-regular space. Then for a function $f:(X,m)\to(Y,l)$ the following conditions are equivalent:

(1) f is weakly (m,l)-open;

(2) For each μ-open set A in X, f(A) is l-open in Y;

(3) For any set B of Y and any μ-closed set A in X containing $f^{-1}(B)$, there exists a l-closed set F in Y containing B such that $f^{-1}(F)\subseteq A$.

Proof. (1) ⇒ (2). Let A be a μ-open set in X. Since X is μ-regular, by Theorem 3.1.9, f is (m,l)-open and A is μ-open. Therefore f(A) is l-open in Y.

(2) ⇒ (3). Let B be any set in Y and A a μ-closed set in X such that $f^{-1}(B)\subseteq A$. Since $X-A$ is μ-open in X, by (2), $f(X-A)$ is l-open in Y. Let $F=Y-f(X-A)$. Then F is μ-closed and $B\subseteq F$. Now $f^{-1}(F)=f^{-1}(Y-f(X-A))=X-f^{-1}(f(X-A))\subseteq A$.

(3) ⇒ (1). Let B be any set in Y. Let $A=c_m(f^{-1}(B))$. Since X is m-regular, A is a μ-closed set in X and $f^{-1}(B)\subseteq A$. Then there exists a l-closed set F in Y containing B such that $f^{-1}(F)\subseteq A$. Since F is l-closed, $f^{-1}(c_l(B))\subseteq f^{-1}(F)\subseteq A=c_m(f^{-1}(B))$. Therefore by **Theorem 3.1.6,** f is weakly (m,l)-open.

3.2. Weakly (m,l)-closed functions

Definition 3.2.1. A functions $f:(X,m)\to(Y,l)$ is said to be (m,l)-closed if for each μ-closed set F of X, f(F) is l-closed.

Definition 3.2.2. A functions $f:(X,m)\to(Y,l)$ is said to be weakly (m,l)-closed if $c_l(f(i_m(F)))\subseteq f(F)$ for each μ-closed set F of X.

Theorem 3.2.3. For a function $f:(X,m)\to(Y,l)$, the following conditions are equivalent:

(1) f is weakly (m,l)-closed;

(2) $c_l(f(U))\subseteq f(c_m(U))$ for each μ-open set U in X.

Proof. (1) ⇒ (2). Let U be a μ-open set in X. Since $c_m(U)$ is a μ-closed set and $U\subseteq i_m(c_m(U))$, we have $c_l(f(U))$ $\subseteq c_l(f(i_m(c_m(U))))\subseteq f(c_m(U))$.

(2) ⇒ (1). Let F be a closed set of X. Then, we have $c_l(f(i_m(F)))\subseteq f(c_m(i_m(F)))\subseteq f(c_m(F))=f(F)$ and hence f is weakly (m,l)-closed.

Corollary 3.2.4. A bijective function $f:(X,m)\to(Y,l)$ is weakly (m,l)-open if and only if f is weakly (m,l)-closed.

Proof. This is an immediate consequence of Theorems 3.1.8 and 3.2.3.

Theorem 3.2.5. For a function $f:(X,m)\to(Y,l)$, the following conditions are equivalent:

(1) f is weakly (m,l)-closed;

(2) $c_l(f(U))\subseteq f(c_m(U))$ for each μ-regular open set U in X;

(3) For each subset F in Y and each μ-open set U in X with $f^{-1}(F)\subseteq U$, there exists a l-open set A in Y with $F\subseteq A$ and $f^{-1}(A)\subseteq c_m(U)$;

(4) For each point y in Y and each μ-open set U in X with $f^{-1}(y)\subseteq U$, there exists a l-open set A in Y containing y and $f^{-1}(A)\subseteq c_m(U)$.

Proof. (1) ⇒ **(2)**. This is clear by Theorem 3.2.3.

(2) ⇒ (3). Let F be a subset of Y and U an μ-open set in X with $f^{-1}(F)\subseteq U$. Then $f^{-1}(F)\cap c_m(X-c_m(U))=f$ and consequently, $F\cap f(c_m(X-c_m(U)))=f$. Since $X-c_m(U)$ is μ-regular open, $F\cap c_l(f(X-c_m(U)))=f$. Let $A=Y-c_l(f(X-c_m(U)))$. Then A is a μ-open set with $F\subseteq A$ and we have $f^{-1}(A)\subseteq X-f^{-1}(c_l(f(X-c_m(U))))\subseteq X-f^{-1}f(X-c_m(U))\subseteq c_m(U)$.

(3) ⇒ (4). This is obvious.

(4) ⇒ (1). Let F be μ-closed in X and let y ϵ Y − f(F). Since $f^{-1}(y) \subseteq X - F$, by (4) there exists a l -open set A in Y with $y \in A$ and $f^{-1}(A) \subseteq c_m(X - F) = X - i_m(F)$. Therefore $A \cap f\left(i_m(F)\right) = f$, so that $y \notin c_l\left(f\left(i_m(F)\right)\right)$. Thus $c_l\left(f\left(i_m(F)\right)\right) \subseteq f(F)$. Hence f is weakly (m, l)-closed.

Theorem 3.2.6. If $f:(X, m) \to (Y, l)$ is a bijective weakly (m, l)-closed function, then for every subset F in Y and every μ-open set U in X with $f^{-1}(F) \subseteq U$, there exists a l -closed set B in Y such that $F \subseteq B$ and $f^{-1}(B) \subseteq c_m(U)$.

Proof. Let F be a subset of Y and U be an μ-open subset of X with $f^{-1}(F) \subseteq U$. But $B = c_l\left(f\left(i_m\left(c_m(U)\right)\right)\right)$. Then B is a l -closed set in Y such that $F \subseteq B$, since $F \subseteq f(U) \subseteq f\left(i_m\left(c_m(U)\right)\right)$ $\subseteq c_l\left(f\left(i_m\left(_m(U)\right)\right)\right) = B$. Since f is weakly (m, l)-closed, we have $f^{-1}(B) \subseteq c_m(U)$.

Recall that, a set F in a space X is (X, m)-compact if for each cover of F by μ-open U in X, there is a finite family $U_1, \dots U_n$ in such that $F \subseteq i_m(\cup\{c_m(Ui) : i = 1, \dots, n\})$.

Theorem 3.2.7. If f: X → Y is a weakly (m, l)-closed function with (m, l)-closed fibers, then *f*(F) is l -closed for each θ-compact set *F* in *X*.

Proof. Let F be a θ-compact set and $y \in Y - f(F)$. Then $f^{-1}(y) \cap F = f$ and for each $x \in F$ there is an μ-open $U_X \subseteq X$ with $x \in U_X$ such that $c_m(U_x) \cap f^{-1}(y) = f$. Clearly $\Omega = \{U_x : x \in F\}$ is an μ-open cover of *F*. Since *F* is θ-compact, there is a finite family $U_{x_1}, \dots U_{x_n}$ in Ω such that $F \subseteq i_l(A)$, where $A = \cup\{c_m(U_{x_i}) : i = 1, \dots, n\}$. Since f is weakly (m, l)-closed, there exists a l -open $B \subseteq Y$ with $f^{-1}(y) \subseteq f^{-1}(B) \subseteq c_m(X - A)$ $= X - i_m(A) \subseteq X - F$. Therefore $y \in B \subseteq Y - f(F)$. $Y - f(F)$ is l -open. This shows that *f*(F) is l -closed.

Theorem 3.2.8. If $f : X \to Y$ is a weakly (m, l)-closed surjection and all pairs of disjoint fibers are μ-strongly separated, then Y is $m - T_2$.

Proof. Let y and z be two points in Y. Let U and V be μ-open sets in X such that $f^{-1}(y) \in U$ and $f^{-1}(z) \in V$ with $c_m(U) \cap c_m(V) = f$.

Since f is weakly (m, l)-closed, there are l-open sets F and B in Y such that $y \in F$ and $z \in B$, $f^{-1}(F) \subseteq c_m(U)$ and $f^{-1}(B) \subseteq c_m(V)$.

Therefore $F \cap B = f$, because $c_m(U) \cap c_m(V) = f$ and f is surjective. Then Y is $m - T_2$.

3.3. Totally (m, l)-Continuous Functions

In this section we introduce the totally (m, l)-continuous functions and strongly (m, l)-continuous and investigate charactrization and relationships among such functions.

Definition 3.3.1. Let (X, m) and (Y, l) be two a generalized topological spaces. A function $f : (X, m) \to (Y, l)$ is called a totally (m, l)-continuous functions if the inverse image of each l-open set in Y is μ-clopen in X.

Theorem 3.3.2. Let (X, m) and (Y, l) be two a generalized topological spaces. A function $f : (X, m) \to (Y, l)$ is a totally (m, l)-continuous surjection and (X, m) is μ-connected, then (Y, l) is an indiscrete space.

Proof. Suppose that (Y, l) is not an indiscrete space and let V be a proper nonempty open subset of (Y, l). Since f is a totally (m, l)-continuous function, then $f^{-1}(V)$ is a proper nonempty μ-clopen subset of X. Therefore $X = f^{-1}(V) \cup (X - f^{-1}(V))$ and X is a union of two nonempty disjoint μ-open sets, which is a contradiction. Therefore X must be an indiscrete space.

Definition 3.3.3. Let (X, m) be a generalized topological spaces. Then X is called a μ-T_0 space if for every two distinct points x and y in X there exists a μ-open set U that contains only one of the point x and y.

Theorem 3.3.4. Let (X, m) and (Y, l) be two a generalized topological spaces. The generalized topological space (X, m) is

μ-connected if and only if every totally (m, l)-continuous function from is a constant map. (Y, l)-T_0-space l (X, m) into any

Proof. (⇒). Suppose that $f:(X,m)\to(Y,l)$ is a totally (m,l)-continuous function, where (Y,l) is a l-T_0-space. Assume that f is not constant and $x, y \in X$ such that $f(x)\neq f(y)$. Since (Y,l) is l-T_0, and $f(x)$ and $f(y)$ are distinct points in Y, then there is an l-open set V in (Y,l) containing only one of the points $f(x)$ and $f(y)$. We take the case $f(x)\in V$ and $f(y)\notin V$. The proof of the other case is similar. Since f is totally (m,l)-continuous function, $f^{-1}(V)$ is a μ-clopen subset of X and $x\in f^{-1}(V)$, $y\notin f^{-1}(V)$. Since $X = f^{-1}(V)\cup(X - f^{-1}(V))$, X is a union of two nonempty disjoint μ-open subset of X. Thus (X,m) is not μ-connected, which is a contradiction.

(⇒). Suppose that (X,m) is not μ-connected, then there is a proper nonempty μ-clopen subset A of X. Let $Y=\{a,b\}$ and $l=\{Y,f,\{a\},\{b\}\}$, define $f:(X,m)\to(Y,l)$ by $f(x)=a$ for each $x\in A$ and $f(x)=b$ for $x\in X-A$. Clearly f is not constant and totally (m,l)-continuous where Y is l-T_0, and thus we have a contradiction.

Theorem 3.3.5. Let (X,m) and (Y,l) be two a generalized topological spaces. A function $f:(X,m)\to(Y,l)$ is a totally (m,l)-continuous injection. If Y is l-T_0 then (X,m) is μ-T_2.

Proof. Let $x, y\in X$ with $x\neq y$. Since f is injection, $f(x)\neq f(y)$. Since Y is l-T_0, there exists an l-open subset V of Y containing $f(x)$ but not $f(y)$, or containing $f(y)$ but not $f(x)$. Thus for the case we have, $x\in f^{-1}(V)$ and $y\notin f^{-1}(V)$. Since f is a totally (m,l)-continuous and V is an l-open subset of Y, $f^{-1}(V)$ and $X-f^{-1}(V)$ are disjoint μ-clopen subset of X containing x and y respectively. The second case is proved in the same way. Thus X is μ-T_2.

Definition 3.3.6. Let (X, m) and (Y, l) be two a generalized topological spaces. A function $f:(X, m) \to (Y, l)$ is called strongly (m, l)-continuous function if the inverse image of every subset of Y is a μ-clopen subset of X.

Remark 3.3.7. Every strongly (m, l)-continuous function is totally (m, l)-continuous, but the converse need not be true as the following example shows.

Example 3.3.8. Let $X = \{a, b, c\}$ and $t = \{X, f, \{a\}, \{b, c\}\}$ be a topology on X. The generalized topological spaces μ is define as follows: $m = \{f, \{a\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Let $f:(X, m) \to (Y, l)$ be the identity function, then f is totally (m, l)-continuous but is not strongly (m, l)-continuous.

Definition 3.3.9. Let (X, m) be a generalized topological spaces. Then X is called a μ-T_1 space if for every two distinct point x and y in X there exists two μ-open set U and V such $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Theorem 3.3.10. Every totally (m, l)-continuous function into a l-T_1-space is strongly (m, l)-continuous.

Proof. Let $f:(X, m) \to (Y, l)$ be a totally (m, l)-continuous function and Y be a l-T_1-space. Then the singletons in Y are l-closed sets. Hence $f^{-1}(B)$ is μ-clopen in X for every subset B of Y. therefore, f is strongly (m, l)-continuous.

3.4. Slightly (m, l)-Continuous Functions.

In this section we introduce the slightly (m, l)-continuous functions and (m, l)-continuous and investigate charactrization and relationships among such functions.

Definition 3.4.1. Let (X, m) and (Y, l) be two a generalized topological spaces. A function $f:(X, m) \to (Y, l)$ is called a slightly (m, l)-continuous function at a point $x \in X$ if for each l-clopen subset V in Y containing $f(x)$, there exists μ-open subset U in X

containing x such that $f(U)\subseteq V$. The function f is said to be slightly (m,l)-continuous if it has this property at each point of X.

Theorem 3.4.2. Let (X,m) and (Y,l) be two a generalized topological spaces. The following statements are equivalent for a function $f:(X,m)\to(Y,l)$:

slightly (m,l)-continuous; (1) f is

(2) for every l-clopen set $V\subseteq Y$, $f^{-1}(V)$ is μ-open;

(3) for every l-clopen set $V\subseteq Y$, $f^{-1}(V)$ is μ-closed;

(4) for every l-clopen set $V\subseteq Y$, $f^{-1}(V)$ is μ-clopen.

Proof. (1)$\Rightarrow$(2). Let V be l-clopen subset of Y and let $x\in f^{-1}(V)$. Since f is slightly (m,l)-continuous, by (1) there exists a μ-open set U_x in X containing x such that $f(U_x)\subseteq V$, hence $U_x\subseteq f^{-1}(V)$. We obtain that $f^{-1}(V)=\cup\{U_x : x\in f^{-1}(V)\}$. Thus, $f^{-1}(V)$ is μ-open.

(2)$\Rightarrow$(3). Let V be l-clopen subset of Y. Then Y-V is l-clopen. By (2) $f^{-1}(Y-V)=X-f^{-1}(V)$ is μ-open. Thus $f^{-1}(V)$ is μ-closed.

(3)$\Rightarrow$(4). It can be shown easily.

(4)$\Rightarrow$(1). Let $x\in X$ and V be l-clopen subset in Y with $f(x)\in V$. Let $U=f^{-1}(V)$. By assumption U is μ-clopen and so μ-open. Also $x\in U$ and $f(U)\subseteq V$.

Theorem 3.4.3. Let (X,m) and (Y,l) be two a generalized topological spaces. A function $f:(X,m)\to(Y,l)$ is slightly (m,l)-continuous surjective function and (X,m) is μ-connected, then Y is l-connected.

Proof. Suppose that Y is not a l-connected space. Then there exist nonempty disjoint l-open sets U and V such that $Y=U\cup V$. Therefore, U and V are l-clopen sets in Y. Since f is slightly (m,l)-continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are μ-open in X. Moreover, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint and $X=f^{-1}(U)\cup f^{-1}(V)$. Since f is surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty. Therefore, X is not μ-connected. This is a contradiction and hence Y is l-connected.

Corollary 3.4.4. Let (X, m) and (Y, l) be two a generalized topological spaces. A function $f:(X,m)\to(Y,l)$. The inverse image of a l-disconnected space under a slightly (m,l)-continuous surjection is μ-disconnected.

Proof. Suppose that Y is a l-disconnected space. Then there exists nonempty disjoint l-open sets U and V such that $Y = U \cup V$. Therefore, U and V are l-clopen sets in Y. Since f slightly (m,l)-continuous then $f^{-1}(U)$ and $f^{-1}(V)$ are μ-clopen sets in X. Moreover, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty and $X = f^{-1}(U)\cup f^{-1}(V)$. Then X is μ-disconnected.

Definition 3.4.5. Let (X, m) be a generalized topological spaces X is said to be:

(1) μ-Locally indiscrete if every μ-open set of X is μ-clopen in X,

(2) μ-0-dimensional is its generalized topological spaces has a base consisting of μ-clopen sets.

Definition 3.4.6. Let (X, m) and (Y, l) be two a generalized topological spaces. A function $f:(X,m)\to(Y,l)$ is called a (m,l)-continuous if the inverse image of l-open sets is μ-open.

Theorem 3.4.7. Let (X, m) and (Y, l) be two a generalized topological spaces. A function $f:(X,m)\to(Y,l)$ is slightly (m,l)-continuous function and Y is l-0-dimensional, then f is (m,l)-continuous

Proof. Let $x \in X$ and $V \subseteq Y$ be any l-open set containing $f(X)$. Since Y is l-0-dimensional, there exists a l-clopen set U containing $f(X)$ such that $V \subseteq Y$. But f is slightly (m,l)- continuous then there exists a μ-open set G containing x such that $f(X)\in f(G)\subseteq U \subseteq V$. Hence f is (m,l)-continuous.

Theorem 3.4.8. Let (X, m) and (Y, l) be two a generalized topological spaces. A function $f:(X,m)\to(Y,l)$ is slightly (m,l)-continuous function and Y is l-Locally indiscrete, then f is (m,l)-continuous.

Proof. Let V be any l-open set of Y. Since Y is l-Locally indiscrete, V is l-clopen and hence $f^{-1}(V)$ is μ-open in X. Therefore, f is (m, l)-continuous.

Theorem 3.4.9. Let (X, m) and (Y, l) be two a generalized topological spaces. A function $f:(X, m)\rightarrow(Y, l)$ is slightly (m, l)-continuous injection and Y is l-0-dimensional. If Y is l-T_1 (resp. l-T_2), then X is μ-T_1 (resp.μ-T_2).

Proof. We prove the second statement, the proof of the first being analogous. Let Y be l-T_2. Since f is injection, for any pair of distinct points $x, y \in X$ we have $f(x) \neq f(y)$. Since Y is l-T_2, there exist l-open sets V_1, V_2 in Y such that $f(x) \in V_1$, $f(y) \notin V_2$ and $V_1 \cap V_2 = f$. Since Y is l-0-dimensional, there exist l-clopen sets U_1, U_2 in Y such that $f(x) \in U_1 \subseteq V_1$ and $f(y) \in U_2 \subseteq V_2$. Consequently $x \in f^{-1}(U_1) \subseteq f^{-1}(V_1)$ and $y \in f^{-1}(U_2) \subseteq f^{-1}(V_2)$. Since f is slightly (m, l)-continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are μ-open sets and this implies that X is μ-T_2.

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